Exercise 5.1

Question 1:
Prove that the function \( f(x) = 5x - 3 \) is continuous at \( x = 0 \), at \( x = -3 \) and at \( x = 5 \).

Answer

The given function is \( f(x) = 5x - 3 \)

At \( x = 0 \), \( f(0) = 5 \times 0 - 3 = 3 \)

\[ \lim_{{x \to 0}} f(x) = \lim_{{x \to 0}} (5x - 3) = 5 \times 0 - 3 = -3 \]

\[ \therefore \lim_{{x \to 0}} f(x) = f(0) \]

Therefore, \( f \) is continuous at \( x = 0 \)

At \( x = -3 \), \( f(-3) = 5 \times (-3) - 3 = -18 \)

\[ \lim_{{x \to -3}} f(x) = \lim_{{x \to -3}} (5x - 3) = 5 \times (-3) - 3 = -18 \]

\[ \therefore \lim_{{x \to -3}} f(x) = f(-3) \]

Therefore, \( f \) is continuous at \( x = -3 \)

At \( x = 5 \), \( f(5) = 5 \times 5 - 3 = 25 - 3 = 22 \)

\[ \lim_{{x \to 5}} f(x) = \lim_{{x \to 5}} (5x - 3) = 5 \times 5 - 3 = 22 \]

\[ \therefore \lim_{{x \to 5}} f(x) = f(5) \]

Therefore, \( f \) is continuous at \( x = 5 \)

Question 2:
Examine the continuity of the function \( f(x) = 2x^2 - 1 \) at \( x = 3 \).

Answer

The given function is \( f(x) = 2x^2 - 1 \)

At \( x = 3 \), \( f(3) = 2 \times 3^2 - 1 = 17 \)

\[ \lim_{{x \to 3}} f(x) = \lim_{{x \to 3}} (2x^2 - 1) = 2 \times 3^2 - 1 = 17 \]

\[ \therefore \lim_{{x \to 3}} f(x) = f(3) \]

Thus, \( f \) is continuous at \( x = 3 \)
Question 3:
Examine the following functions for continuity.

(a) \( f(x) = x - 5 \)  
(b) \( f(x) = \frac{1}{x - 5}, x \neq 5 \)

(c) \( f(x) = \frac{x^2 - 25}{x + 5}, x \neq -5 \)  
(d) \( f(x) = |x - 5| \)

Answer

(a) The given function is \( f(x) = x - 5 \)
It is evident that \( f \) is defined at every real number \( k \) and its value at \( k \) is \( k - 5 \).

It is also observed that, \( \lim_{x \to k} f(x) = \lim_{x \to k} (x - 5) = k - 5 = f(k) \)
\( \therefore \lim_{x \to k} f(x) = f(k) \)
Hence, \( f \) is continuous at every real number and therefore, it is a continuous function.

(b) The given function is \( f(x) = \frac{1}{x - 5}, x \neq 5 \)
For any real number \( k \neq 5 \), we obtain
\[ \lim_{x \to k} f(x) = \lim_{x \to k} \frac{1}{x - 5} = \frac{1}{k - 5} \]
Also, \( f(k) = \frac{1}{k - 5} \)  \( \text{ (As } k \neq 5 \) \)
\( \therefore \lim_{x \to k} f(x) = f(k) \)
Hence, \( f \) is continuous at every point in the domain of \( f \) and therefore, it is a continuous function.

(c) The given function is \( f(x) = \frac{x^2 - 25}{x + 5}, x \neq -5 \)
For any real number \( c \neq -5 \), we obtain
\[
\lim_{x \to c} f(x) = \lim_{x \to c} \frac{x^2 - 25}{x + 5} = \lim_{x \to c} \frac{(x + 5)(x - 5)}{x + 5} = \lim_{x \to c} (x - 5) = (c - 5)
\]

Also, \( f(c) = \frac{(c + 5)(c - 5)}{c + 5} = (c - 5) \) (as \( c \neq -5 \))

\[
\therefore \lim_{x \to c} f(x) = f(c)
\]

Hence, \( f \) is continuous at every point in the domain of \( f \) and therefore, it is a continuous function.

\[
f(x) = |x - 5| = \begin{cases} 
5 - x & \text{if } x < 5 \\
5 - x & \text{if } x \geq 5
\end{cases}
\]

(d) The given function is

This function \( f \) is defined at all points of the real line.

Let \( c \) be a point on a real line. Then, \( c < 5 \) or \( c = 5 \) or \( c > 5 \)

Case I: \( c < 5 \)

Then, \( f(c) = 5 - c \)

\[
\lim_{x \to c} f(x) = \lim_{x \to c} (5 - x) = 5 - c
\]

\[
\therefore \lim_{x \to c} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all real numbers less than 5.

Case II : \( c = 5 \)

Then, \( f(c) = f(5) = (5 - 5) = 0 \)

\[
\lim_{x \to 5} f(x) = \lim_{x \to 5} (5 - x) = (5 - 5) = 0
\]

\[
\lim_{x \to 5} f(x) = \lim_{x \to 5} (x - 5) = 0
\]

\[
\therefore \lim_{x \to 5} f(x) = \lim_{x \to c} f(x) = f(c)
\]

Therefore, \( f \) is continuous at \( x = 5 \)

Case III: \( c > 5 \)

Then, \( f(c) = f(5) = c - 5 \)

\[
\lim_{x \to c} f(x) = \lim_{x \to c} (x - 5) = c - 5
\]

\[
\therefore \lim_{x \to c} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all real numbers greater than 5.

Hence, \( f \) is continuous at every real number and therefore, it is a continuous function.
Question 4:
Prove that the function \( f(x) = x^n \) is continuous at \( x = n \), where \( n \) is a positive integer.

Answer
The given function is \( f(x) = x^n \)
It is evident that \( f \) is defined at all positive integers, \( n \), and its value at \( n \) is \( n^n \).

Then, \( \lim_{x \to n} f(n) = \lim_{x \to n} (x^n) = n^n \)

\( \therefore \lim_{x \to n} f(x) = f(n) \)

Therefore, \( f \) is continuous at \( n \), where \( n \) is a positive integer.

Question 5:
Is the function \( f \) defined by
\[
f(x) = \begin{cases} 
 x, & \text{if } x \leq 1 \\
 5, & \text{if } x > 1 
\end{cases}
\]
continuous at \( x = 0 \)? At \( x = 1 \)? At \( x = 2 \)?

Answer
The given function \( f \) is
At \( x = 0 \),
It is evident that \( f \) is defined at \( 0 \) and its value at \( 0 \) is \( 0 \).

Then, \( \lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0 \)

\( \therefore \lim_{x \to 0} f(x) = f(0) \)

Therefore, \( f \) is continuous at \( x = 0 \)
At \( x = 1 \),
\( f \) is defined at \( 1 \) and its value at \( 1 \) is \( 1 \).
The left hand limit of \( f \) at \( x = 1 \) is,
\( \lim_{x \to 1^-} f(x) = \lim_{x \to 1} x = 1 \)
The right hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} (5) = 5
\]

\[\therefore \lim_{x \to 1} f(x) \neq \lim_{x \to 1} f(x)\]

Therefore, \( f \) is not continuous at \( x = 1 \)

At \( x = 2 \),
\( f \) is defined at 2 and its value at 2 is 5.

Then, \( \lim_{x \to 2} f(x) = \lim_{x \to 2} (5) = 5 \)

\[\therefore \lim_{x \to 2} f(x) = f(2)\]

Therefore, \( f \) is continuous at \( x = 2 \)

**Question 6:**
Find all points of discontinuity of \( f \), where \( f \) is defined by

\[f(x) = \begin{cases} 2x + 3, & \text{if } x \leq 2 \\ 2x - 3, & \text{if } x > 2 \end{cases}\]

**Answer**

\[f(x) = \begin{cases} 2x + 3, & \text{if } x \leq 2 \\ 2x - 3, & \text{if } x > 2 \end{cases}\]

The given function \( f \) is

It is evident that the given function \( f \) is defined at all the points of the real line.

Let \( c \) be a point on the real line. Then, three cases arise.

(i) \( c < 2 \)
(ii) \( c > 2 \)
(iii) \( c = 2 \)

**Case (i) \( c < 2 \)**

Then, \( f(c) = 2c + 3 \)

\[\lim_{x \to c^-} f(x) = \lim_{x \to c} (2x + 3) = 2c + 3\]

\[\therefore \lim_{x \to c} f(x) = f(c)\]

Therefore, \( f \) is continuous at all points \( x \), such that \( x < 2 \)

**Case (ii) \( c > 2 \)**
Therefore, \( f \) is continuous at all points \( x \), such that \( x > 2 \)

Case (iii) \( c = 2 \)

Then, the left hand limit of \( f \) at \( x = 2 \) is,

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x+3) = 2 \times 2 + 3 = 7
\]

The right hand limit of \( f \) at \( x = 2 \) is,

\[
\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} (2x-3) = 2 \times 2 - 3 = 1
\]

It is observed that the left and right hand limit of \( f \) at \( x = 2 \) do not coincide.

Therefore, \( f \) is not continuous at \( x = 2 \)

Hence, \( x = 2 \) is the only point of discontinuity of \( f \).

**Question 7:**

Find all points of discontinuity of \( f \), where \( f \) is defined by

\[
f(x) = \begin{cases} 
|x|+3, & \text{if } x \leq -3 \\
-2x, & \text{if } -3 < x < 3 \\
6x+2, & \text{if } x \geq 3 
\end{cases}
\]

**Answer**

\[
f(x) = \begin{cases} 
|x|+3 = -x+3, & \text{if } x \leq -3 \\
-2x, & \text{if } -3 < x < 3 \\
6x+2, & \text{if } x \geq 3 
\end{cases}
\]

The given function \( f \) is

The given function \( f \) is defined at all the points of the real line.

Let \( c \) be a point on the real line.

**Case I:**

If \( c < -3 \), then \( f(c) = -c+3 \)

\[
\lim_{x \to c^-} f(x) = \lim_{x \to c^-} (-x+3) = -c+3
\]

\[
\lim_{x \to c^+} f(x) = f(c)
\]

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Therefore, $f$ is continuous at all points $x$, such that $x < -3$

Case II:
If $c = -3$, then $f(-3) = -(3) + 3 = 6$

$$\lim_{{x \to -3}} f(x) = \lim_{{x \to -3}} (-x + 3) = -(3) + 3 = 6$$

$$\lim_{{x \to -3}} f(x) = \lim_{{x \to -3}} (-2x) = -2(-3) = 6$$

\[ \therefore \lim_{{x \to -3}} f(x) = f(-3) \]

Therefore, $f$ is continuous at $x = -3$

Case III:
If $-3 < c < 3$, then $f(c) = -2c$ and $\lim_{{x \to c}} f(x) = \lim_{{x \to c}} (-2x) = -2c$

\[ \therefore \lim_{{x \to c}} f(x) = f(c) \]

Therefore, $f$ is continuous in $(-3, 3)$.

Case IV:
If $c = 3$, then the left hand limit of $f$ at $x = 3$ is,

$$\lim_{{x \to 3}} f(x) = \lim_{{x \to 3}} (-2x) = -2 \times 3 = -6$$

The right hand limit of $f$ at $x = 3$ is,

$$\lim_{{x \to 3}} f(x) = \lim_{{x \to 3}} (6x + 2) = 6 \times 3 + 2 = 20$$

It is observed that the left and right hand limit of $f$ at $x = 3$ do not coincide.

Therefore, $f$ is not continuous at $x = 3$

Case V:
If $c > 3$, then $f'(c) = 6c + 2$ and $\lim_{{x \to c}} f(x) = \lim_{{x \to c}} (6x + 2) = 6c + 2$

\[ \therefore \lim_{{x \to c}} f(x) = f(c) \]

Therefore, $f$ is continuous at all points $x$, such that $x > 3$

Hence, $x = 3$ is the only point of discontinuity of $f$.

**Question 8:**
Find all points of discontinuity of $f$, where $f$ is defined by
The given function \( f \) is defined at all the points of the real line. Let \( c \) be a point on the real line.

Case I:
If \( c < 0 \), then \( f(c) = -1 \)
\[
\lim_{x \to c} f(x) = \lim_{x \to c} (-1) = -1
\]
\[
\therefore \lim_{x \to c} f(x) = f(c)
\]
Therefore, \( f \) is continuous at all points \( x < 0 \)

Case II:
If \( c = 0 \), then the left hand limit of \( f \) at \( x = 0 \) is,
\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1) = -1
\]
The right hand limit of \( f \) at \( x = 0 \) is,
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1) = 1
\]
It is observed that the left and right hand limit of \( f \) at \( x = 0 \) do not coincide.

Therefore, \( f \) is not continuous at \( x = 0 \)

Case III:
If \( c > 0 \), then \( f(c) = 1 \)
\[
\lim_{x \to c} f(x) = \lim_{x \to c} \left( \frac{x}{|x|} \right) = 1
\]
\[
\therefore \lim_{x \to c} f(x) = f(c)
\]
Therefore, \( f \) is continuous at all points \( x \), such that \( x > 0 \)
Hence, \( x = 0 \) is the only point of discontinuity of \( f \).

**Question 9:**
Find all points of discontinuity of \( f \), where \( f \) is defined by
\[
f(x) = \begin{cases} 
\frac{x}{|x|}, & \text{if } x < 0 \\
-1, & \text{if } x \geq 0 
\end{cases}
\]

**Answer**
\[
f(x) = \begin{cases} 
\frac{x}{|x|}, & \text{if } x < 0 \\
-1, & \text{if } x \geq 0 
\end{cases}
\]
The given function \( f \) is

It is known that, \( x < 0 \Rightarrow |x| = -x \)
Therefore, the given function can be rewritten as
\[
f(x) = \begin{cases} 
\frac{x}{|x|} = \frac{x}{-x} = -1, & \text{if } x < 0 \\
-1, & \text{if } x \geq 0 
\end{cases}
\]
\[\Rightarrow f(x) = -1 \text{ for all } x \in \mathbb{R} \]

Let \( c \) be any real number. Then, \( \lim_{x \to c} f(x) = \lim_{x \to c} (-1) = -1 \)

Also,
\[f(c) = -1 = \lim_{x \to c} f(x)\]
Therefore, the given function is a continuous function.
Hence, the given function has no point of discontinuity.

**Question 10:**
Find all points of discontinuity of \( f \), where \( f \) is defined by
The given function \( f \) is defined at all the points of the real line.

Let \( c \) be a point on the real line.

Case I:

If \( c < 1 \), then \( f(c) = c^2 + 1 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (x^2 + 1) = c^2 + 1 \)

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x < 1 \)

Case II:

If \( c = 1 \), then \( f(1) = 1 + 1 = 2 \)

The left hand limit of \( f \) at \( x = 1 \) is,

\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = 1^2 + 1 = 2 \]

The right hand limit of \( f \) at \( x = 1 \) is,

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = 1 + 1 = 2 \]

\[ \therefore \lim_{x \to 1} f(x) = f(1) \]

Therefore, \( f \) is continuous at \( x = 1 \)

Case III:

If \( c > 1 \), then \( f(c) = c + 1 \)

\[ \lim_{x \to c} f(x) = \lim_{x \to c} (x + 1) = c + 1 \]

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 1 \)

Hence, the given function \( f \) has no point of discontinuity.

Question 11:

Find all points of discontinuity of \( f \), where \( f \) is defined by
The given function $f$ is defined at all the points of the real line. Let $c$ be a point on the real line.

Case I:

If $c < 2$, then $f(c) = c^3 - 3$ and 
$$\lim_{x \to c} f(x) = \lim_{x \to c} (x^3 - 3) = c^3 - 3$$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, $f$ is continuous at all points $x$, such that $x < 2$

Case II:

If $c = 2$, then $f(c) = f(2) = 2^3 - 3 = 5$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x^3 - 3) = 2^3 - 3 = 5$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x^2 + 1) = 2^2 + 1 = 5$$

$\therefore \lim_{x \to 2} f(x) = f(2)$

Therefore, $f$ is continuous at $x = 2$

Case III:

If $c > 2$, then $f(c) = c^2 + 1$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x^2 + 1) = c^2 + 1$$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, $f$ is continuous at all points $x$, such that $x > 2$

Thus, the given function $f$ is continuous at every point on the real line.

Hence, $f$ has no point of discontinuity.

**Question 12:**
Find all points of discontinuity of $f$, where $f$ is defined by

$$f(x) = \begin{cases} 
  x^3 - 3, & \text{if } x \leq 2 \\
  x^2 + 1, & \text{if } x > 2 
\end{cases}$$
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\[ f(x) = \begin{cases} 
  x^{10} - 1, & \text{if } x \leq 1 \\
  x^2, & \text{if } x > 1
\end{cases} \]

**Answer**

\[ f(x) = \begin{cases} 
  x^{10} - 1, & \text{if } x \leq 1 \\
  x^2, & \text{if } x > 1
\end{cases} \]

The given function \( f \) is defined at all the points of the real line.

Let \( c \) be a point on the real line.

**Case I:**

If \( c < 1 \), then \( f(c) = c^{10} - 1 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (x^{10} - 1) = c^{10} - 1 \)

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x < 1 \)

**Case II:**

If \( c = 1 \), then the left hand limit of \( f \) at \( x = 1 \) is,

\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1} (x^{10} - 1) = 1^{10} - 1 = 1 - 1 = 0 \]

The right hand limit of \( f \) at \( x = 1 \) is,

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1} (x^2) = 1^2 = 1 \]

It is observed that the left and right hand limit of \( f \) at \( x = 1 \) do not coincide.

Therefore, \( f \) is not continuous at \( x = 1 \)

**Case III:**

If \( c > 1 \), then \( f(c) = c^2 \)

\[ \lim_{x \to c} f(x) = \lim_{x \to c} (x^2) = c^2 \]

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 1 \)

Thus, from the above observation, it can be concluded that \( x = 1 \) is the only point of discontinuity of \( f \).

**Question 13:**

Is the function defined by

\[ f(x) = \begin{cases} 
  x^{10} - 1, & \text{if } x \leq 1 \\
  x^2, & \text{if } x > 1
\end{cases} \]
f(x) = \begin{cases} 
  x + 5, & \text{if } x \leq 1 \\
  x - 5, & \text{if } x > 1
\end{cases}

a continuous function?

Answer

\[ f(x) = \begin{cases} 
  x + 5, & \text{if } x \leq 1 \\
  x - 5, & \text{if } x > 1
\end{cases} \]

The given function is defined at all the points of the real line. Let c be a point on the real line.

Case I:

If \( c < 1 \), then \( f(c) = c + 5 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (x + 5) = c + 5 \)

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x < 1 \)

Case II:

If \( c = 1 \), then \( f(1) = 1 + 5 = 6 \)

The left hand limit of \( f \) at \( x = 1 \) is,

\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 5) = 1 + 5 = 6 \]

The right hand limit of \( f \) at \( x = 1 \) is,

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x - 5) = 1 - 5 = -4 \]

It is observed that the left and right hand limit of \( f \) at \( x = 1 \) do not coincide. Therefore, \( f \) is not continuous at \( x = 1 \)

Case III:

If \( c > 1 \), then \( f(c) = c - 5 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (x - 5) = c - 5 \)

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 1 \)

Thus, from the above observation, it can be concluded that \( x = 1 \) is the only point of discontinuity of \( f \).

Question 14:
Discuss the continuity of the function \( f \), where \( f \) is defined by

\[
\begin{cases}
3, & \text{if } 0 \leq x \leq 1 \\
4, & \text{if } 1 < x < 3 \\
5, & \text{if } 3 \leq x \leq 10
\end{cases}
\]

Answer

\[
\begin{cases}
3, & \text{if } 0 \leq x \leq 1 \\
4, & \text{if } 1 < x < 3 \\
5, & \text{if } 3 \leq x \leq 10
\end{cases}
\]

The given function is defined at all points of the interval \([0, 10]\).

Let \( c \) be a point in the interval \([0, 10]\).

Case I:

If \( 0 \leq c < 1 \), then \( f(c) = 3 \) and \( \lim_{{x \to c}} f(x) = \lim_{{x \to c}} (3) = 3 \)

\[ \therefore \lim_{{x \to c}} f(x) = f(c) \]

Therefore, \( f \) is continuous in the interval \([0, 1)\).

Case II:

If \( c = 1 \), then \( f(3) = 3 \)

The left hand limit of \( f \) at \( x = 1 \) is,

\[ \lim_{{x \to 1^-}} f(x) = \lim_{{x \to 1^-}} (3) = 3 \]

The right hand limit of \( f \) at \( x = 1 \) is,

\[ \lim_{{x \to 1^+}} f(x) = \lim_{{x \to 1^+}} (4) = 4 \]

It is observed that the left and right hand limits of \( f \) at \( x = 1 \) do not coincide. Therefore, \( f \) is not continuous at \( x = 1 \)

Case III:

If \( 1 < c < 3 \), then \( f(c) = 4 \) and \( \lim_{{x \to c}} f(x) = \lim_{{x \to c}} (4) = 4 \)

\[ \therefore \lim_{{x \to c}} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points of the interval \((1, 3)\).

Case IV:

If \( c = 3 \), then \( f(c) = 5 \)
The left hand limit of \( f \) at \( x = 3 \) is,
\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (4) = 4
\]

The right hand limit of \( f \) at \( x = 3 \) is,
\[
\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (5) = 5
\]

It is observed that the left and right hand limits of \( f \) at \( x = 3 \) do not coincide.

Therefore, \( f \) is not continuous at \( x = 3 \)

Case V:

If \( 3 < c \leq 10 \), then \( f(c) = 5 \) and
\[
\lim_{x \to c^-} f(x) = \lim_{x \to c} (5) = 5
\]
\[
\lim_{x \to c^+} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all points of the interval \((3, 10]\).

Hence, \( f \) is not continuous at \( x = 1 \) and \( x = 3 \)

**Question 15:**

Discuss the continuity of the function \( f \), where \( f \) is defined by
\[
f(x) = \begin{cases} 
2x, & \text{if } x < 0 \\
0, & \text{if } 0 \leq x \leq 1 \\
4x, & \text{if } x > 1 
\end{cases}
\]

Answer

\[
f(x) = \begin{cases} 
2x, & \text{if } x < 0 \\
0, & \text{if } 0 \leq x \leq 1 \\
4x, & \text{if } x > 1 
\end{cases}
\]

The given function is

The given function is defined at all points of the real line.

Let \( c \) be a point on the real line.

Case I:

If \( c < 0 \), then \( f(c) = 2c \)
\[
\lim_{x \to c^-} f(x) = \lim_{x \to c^-} (2x) = 2c
\]
\[
\therefore \lim_{x \to c} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all points \( x \), such that \( x < 0 \)
Case II:
If \( c = 0 \), then \( f(c) = f(0) = 0 \)

The left hand limit of \( f \) at \( x = 0 \) is,
\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x) = 2 \times 0 = 0
\]

The right hand limit of \( f \) at \( x = 0 \) is,
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (0) = 0
\]
\[
\therefore \lim_{x \to 0} f(x) = f(0)
\]

Therefore, \( f \) is continuous at \( x = 0 \)

Case III:
If \( 0 < c < 1 \), then \( f(x) = 0 \) and \( \lim_{x \to c^-} f(x) = \lim_{x \to c^+} (0) = 0 \)
\[
\therefore \lim_{x \to c^-} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all points of the interval \((0, 1)\).

Case IV:
If \( c = 1 \), then \( f(c) = f(1) = 0 \)

The left hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (0) = 0
\]

The right hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x) = 4 \times 1 = 4
\]

It is observed that the left and right hand limits of \( f \) at \( x = 1 \) do not coincide.

Therefore, \( f \) is not continuous at \( x = 1 \)

Case V:
If \( c < 1 \), then \( f(c) = 4c \) and \( \lim_{x \to c^-} f(x) = \lim_{x \to c^+} (4x) = 4c \)
\[
\therefore \lim_{x \to c^-} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 1 \)
Hence, \( f \) is not continuous only at \( x = 1 \)

**Question 16:**
Discuss the continuity of the function \( f \), where \( f \) is defined by
The given function $f$ is defined at all points of the real line.

Let $c$ be a point on the real line.

**Case I:**

If $c < -1$, then $f(c) = -2$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (-2) = -2$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, $f$ is continuous at all points $x$, such that $x < -1$

**Case II:**

If $c = -1$, then $f(c) = f(-1) = -2$

The left hand limit of $f$ at $x = -1$ is,

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-2) = -2$

The right hand limit of $f$ at $x = -1$ is,

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (2x) = 2 \times (-1) = -2$

$\therefore \lim_{x \to -1} f(x) = f(-1)$

Therefore, $f$ is continuous at $x = -1$

**Case III:**

If $-1 < c < 1$, then $f(c) = 2c$

$\lim_{x \to c} f(x) = \lim_{x \to c} (2x) = 2c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, $f$ is continuous at all points of the interval $(-1, 1)$.

**Case IV:**
If \( c = 1 \), then \( f(c) = f(1) = 2 \times 1 = 2 \)

The left hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^{-}} f(x) = \lim_{x \to 1^{-}} (2x) = 2 \times 1 = 2
\]

The right hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^{+}} f(x) = \lim_{x \to 1^{+}} 2 = 2
\]

\[
\therefore \lim_{x \to 1} f(x) = f(c)
\]

Therefore, \( f \) is continuous at \( x = 2 \)

Case V:

If \( c > 1 \), then \( f(c) = 2 \) and \( \lim_{x \to c^{-}} f(x) = \lim_{x \to c^{-}} 2 = 2 \)

\[
\lim_{x \to c} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 1 \)

Thus, from the above observations, it can be concluded that \( f \) is continuous at all points of the real line.

**Question 17:**

Find the relationship between \( a \) and \( b \) so that the function \( f \) defined by

\[
f(x) = \begin{cases} 
  ax + 1, & \text{if } x \leq 3 \\ 
  bx + 3, & \text{if } x > 3 
\end{cases}
\]

is continuous at \( x = 3 \).

Answer

\[
f(x) = \begin{cases} 
  ax + 1, & \text{if } x \leq 3 \\ 
  bx + 3, & \text{if } x > 3 
\end{cases}
\]

The given function \( f \) is

If \( f \) is continuous at \( x = 3 \), then
\[
\lim_{x \to 3} f(x) = \lim_{x \to 3} f(x) = f(3) \quad \text{...(1)}
\]

Also,
\[
\lim_{x \to 3} f(x) = \lim_{x \to 3} (ax + 1) = 3a + 1
\]
\[
\lim_{x \to 3} f(x) = \lim_{x \to 3} (bx + 3) = 3b + 3
\]
\[
f(3) = 3a + 1
\]

Therefore, from (1), we obtain
\[
3a + 1 = 3b + 3 = 3a + 1
\]
\Rightarrow 3a + 1 = 3b + 3
\Rightarrow 3a = 3b + 2
\Rightarrow a = b + \frac{2}{3}

Therefore, the required relationship is given by,
\[
a = b + \frac{2}{3}
\]

**Question 18:**

For what value of \( \lambda \) is the function defined by
\[
f(x) = \begin{cases} 
\lambda \left(x^2 - 2x\right), & \text{if } x \leq 0 \\
4x + 1, & \text{if } x > 0
\end{cases}
\]
continuous at \( x = 0 \)? What about continuity at \( x = 1 \)?

Answer

\[
f(x) = \begin{cases} 
\lambda \left(x^2 - 2x\right), & \text{if } x \leq 0 \\
4x + 1, & \text{if } x > 0
\end{cases}
\]

The given function \( f \) is

If \( f \) is continuous at \( x = 0 \), then
\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)
\]
\Rightarrow \lambda \left(0^2 - 2\times 0\right) = \lim_{x \to 0^+} (4x + 1) = \lambda \left(0^2 - 2\times 0\right)
\Rightarrow \lambda \left(0^2 - 2\times 0\right) = 4 \times 0 + 1 = 0
\Rightarrow 0 = 1 = 0, \text{ which is not possible}

Therefore, there is no value of \( \lambda \) for which \( f \) is continuous at \( x = 0 \)

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At $x = 1$,
\[ f(1) = 4x + 1 = 4 \times 1 + 1 = 5 \]
\[ \lim_{x \to 1} (4x + 1) = 4 \times 1 + 1 = 5 \]
\[ \therefore \lim_{x \to 1} f(x) = f(1) \]
Therefore, for any values of $\lambda$, $f$ is continuous at $x = 1$

**Question 19:**

Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral point.

Here $[x]$ denotes the greatest integer less than or equal to $x$.

**Answer**

The given function is $g(x) = x - [x]$

It is evident that $g$ is defined at all integral points.

Let $n$ be an integer.

Then,
\[ g(n) = n - [n] = n - n = 0 \]

The left hand limit of $f$ at $x = n$ is,
\[ \lim_{x \to n^-} g(x) = \lim_{x \to n^-} (x - [x]) = \lim_{x \to n^-} (x) - \lim_{x \to n^-} [x] = n - (n - 1) = 1 \]

The right hand limit of $f$ at $x = n$ is,
\[ \lim_{x \to n^+} g(x) = \lim_{x \to n^+} (x - [x]) = \lim_{x \to n^+} (x) - \lim_{x \to n^+} [x] = n - n = 0 \]

It is observed that the left and right hand limits of $f$ at $x = n$ do not coincide.

Therefore, $f$ is not continuous at $x = n$

Hence, $g$ is discontinuous at all integral points.

**Question 20:**

Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = p$?

**Answer**

The given function is $f(x) = x^2 - \sin x + 5$
It is evident that \( f \) is defined at \( x = p \)

At \( x = \pi \), \( f(x) = f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5 \)

Consider \( \lim_{x \to \pi} f(x) = \lim_{x \to \pi} (x^2 - \sin x + 5) \)

Put \( x = \pi + h \)

If \( x \to \pi \), then it is evident that \( h \to 0 \)

\[ \lim_{x \to \pi} f(x) = \lim_{x \to \pi} (x^2 - \sin x + 5) \]

\[ = \lim_{h \to 0} [(\pi + h)^2 - \sin(\pi + h) + 5] \]

\[ = \lim_{h \to 0} (\pi^2 + 2\pi h + h^2) - \lim_{h \to 0} \sin(\pi + h) + \lim_{h \to 0} 5 \]

\[ = (\pi + 0)^2 - \lim_{h \to 0} \sin \pi \cosh + \cos \pi \sinh + 5 \]

\[ = \pi^2 - \lim_{h \to 0} \sin \pi \cosh - \lim_{h \to 0} \cos \pi \sinh + 5 \]

\[ = \pi^2 - 0 \times 1 - (-1) \times 0 + 5 \]

\[ = \pi^2 + 5 \]

\[ \therefore \lim_{x \to \pi} f(x) = f(\pi) \]

Therefore, the given function \( f \) is continuous at \( x = \pi \)

**Question 21:**

Discuss the continuity of the following functions.

(a) \( f(x) = \sin x + \cos x \)

(b) \( f(x) = \sin x - \cos x \)

(c) \( f(x) = \sin x \times \cos x \)

**Answer**

It is known that if \( g \) and \( h \) are two continuous functions, then

\( g + h, \ g - h, \) and \( gh \) are also continuous.

It has to proved first that \( g(x) = \sin x \) and \( h(x) = \cos x \) are continuous functions.

Let \( g(x) = \sin x \)

It is evident that \( g(x) = \sin x \) is defined for every real number.

Let \( c \) be a real number. Put \( x = c + h \)

If \( x \to c \), then \( h \to 0 \)
\[ g(c) = \sin c \]
\[ \lim_{x \to c} g(x) = \lim_{x \to c} \sin x \]
\[ = \lim_{h \to 0} \sin (c + h) \]
\[ = \lim_{h \to 0} \left[ \sin c \cos h + \cos c \sin h \right] \]
\[ = \lim_{h \to 0} (\sin c \cos h) + \lim_{h \to 0} (\cos c \sin h) \]
\[ = \sin c \cos 0 + \cos c \sin 0 \]
\[ = \sin c + 0 \]
\[ = \sin c \]
\[ \therefore \lim_{x \to c} g(x) = g(c) \]

Therefore, \( g \) is a continuous function.

Let \( h(x) = \cos x \)

It is evident that \( h(x) = \cos x \) is defined for every real number.

Let \( c \) be a real number. Put \( x = c + h \)

If \( x \to c \), then \( h \to 0 \)

\[ h(c) = \cos c \]
\[ \lim_{x \to c} h(x) = \lim_{x \to c} \cos x \]
\[ = \lim_{h \to 0} \cos (c + h) \]
\[ = \lim_{h \to 0} \left[ \cos c \cos h - \sin c \sin h \right] \]
\[ = \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h \]
\[ = \cos c \cos 0 - \sin c \sin 0 \]
\[ = \cos c \times 1 - \sin c \times 0 \]
\[ = \cos c \]
\[ \therefore \lim_{x \to c} h(x) = h(c) \]

Therefore, \( h \) is a continuous function.

Therefore, it can be concluded that

(a) \( f(x) = g(x) + h(x) = \sin x + \cos x \) is a continuous function

(b) \( f(x) = g(x) - h(x) = \sin x - \cos x \) is a continuous function

(c) \( f(x) = g(x) \times h(x) = \sin x \times \cos x \) is a continuous function
Question 22:
Discuss the continuity of the cosine, cosecant, secant and cotangent functions,
Answer
It is known that if $g$ and $h$ are two continuous functions, then

(i) $\frac{h(x)}{g(x)}$, $g(x) \neq 0$ is continuous

(ii) $\frac{1}{g(x)}$, $g(x) \neq 0$ is continuous

(iii) $\frac{1}{h(x)}$, $h(x) \neq 0$ is continuous

It has to be proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.
Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.
Let $c$ be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$g(c) = \sin c$

$\lim_{x \to c} g(x) = \lim_{x \to c} \sin x$

$= \lim_{h \to 0} \sin(c + h)$

$= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h]$

$= \lim_{h \to 0} (\sin c \cos h) + \lim_{h \to 0} (\cos c \sin h)$

$= \sin c \cos 0 + \cos c \sin 0$

$= \sin c + 0$

$= \sin c$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, $g$ is a continuous function.
Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.
Let $c$ be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$
\[
\lim_{x \to c} h(x) = \lim_{x \to c} \cos x \\
= \lim_{h \to 0} \cos(c + h) \\
= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h] \\
= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h \\
= \cos c \cos 0 - \sin c \sin 0 \\
= \cos c \times 1 - \sin c \times 0 \\
= \cos c
\]

\[
\therefore \lim_{x \to c} h(x) = h(c)
\]

Therefore, \( h(x) = \cos x \) is a continuous function.

It can be concluded that,

\[
\csc x = \frac{1}{\sin x}, \quad \sin x \neq 0 \text{ is continuous}
\]

\[
\Rightarrow \csc x, \quad x \neq n\pi \quad (n \in \mathbb{Z}) \text{ is continuous}
\]

Therefore, cosecant is continuous except at \( x = np, n \in \mathbb{Z} \)

\[
\sec x = \frac{1}{\cos x}, \quad \cos x \neq 0 \text{ is continuous}
\]

\[
\Rightarrow \sec x, \quad x \neq (2n+1)\frac{\pi}{2} \quad (n \in \mathbb{Z}) \text{ is continuous}
\]

Therefore, secant is continuous except at \( x = (2n+1)\frac{\pi}{2} \quad (n \in \mathbb{Z}) \)

\[
\cot x = \frac{\cos x}{\sin x}, \quad \sin x \neq 0 \text{ is continuous}
\]

\[
\Rightarrow \cot x, \quad x \neq n\pi \quad (n \in \mathbb{Z}) \text{ is continuous}
\]

Therefore, cotangent is continuous except at \( x = np, n \in \mathbb{Z} \)

**Question 23:**

Find the points of discontinuity of \( f \), where

\[
f(x) = \begin{cases} 
\frac{\sin x}{x}, & \text{if } x < 0 \\
\frac{x}{x+1}, & \text{if } x \geq 0
\end{cases}
\]
Answer

\[ f(x) = \begin{cases} 
\frac{\sin x}{x}, & \text{if } x < 0 \\
x + 1, & \text{if } x \geq 0 
\end{cases} \]

The given function \( f \) is.

It is evident that \( f \) is defined at all points of the real line.

Let \( c \) be a real number.

Case I:

If \( c < 0 \), then \( f(c) = \frac{\sin c}{c} \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} \left( \frac{\sin x}{x} \right) = \frac{\sin c}{c} \)

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x < 0 \)

Case II:

If \( c > 0 \), then \( f(c) = c + 1 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (x + 1) = c + 1 \)

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 0 \)

Case III:

If \( c = 0 \), then \( f(c) = f(0) = 0 + 1 = 1 \)

The left hand limit of \( f \) at \( x = 0 \) is,

\[ \lim_{x \to 0^-} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1 \]

The right hand limit of \( f \) at \( x = 0 \) is,

\[ \lim_{x \to 0^+} f(x) = \lim_{x \to 0} (x + 1) = 1 \]

\[ \therefore \lim_{x \to 0} f(x) = \lim_{x \to 0} f(x) = f(0) \]

Therefore, \( f \) is continuous at \( x = 0 \)

From the above observations, it can be concluded that \( f \) is continuous at all points of the real line.

Thus, \( f \) has no point of discontinuity.
Question 24:

Determine if $f$ defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a continuous function?

Answer

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

The given function $f$ is

It is evident that $f$ is defined at all points of the real line.

Let $c$ be a real number.

Case I:

If $c \neq 0$, then $f(c) = c^2 \sin \frac{1}{c}$

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left( x^2 \sin \frac{1}{x} \right) = \left( \lim_{x \to c} x^2 \right) \left( \lim_{x \to c} \sin \frac{1}{x} \right) = c^2 \sin \frac{1}{c}$$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, $f$ is continuous at all points $x \neq 0$

Case II:

If $c = 0$, then $f(0) = 0$
Therefore, $f(x)$ is continuous at $x = 0$

From the above observations, it can be concluded that $f$ is continuous at every point of the real line.

Thus, $f$ is a continuous function.

**Question 25:**
Examine the continuity of $f$, where $f$ is defined by

$$f(x) = \begin{cases} 
\sin x - \cos x, & \text{if } x \neq 0 \\
-1 & \text{if } x = 0
\end{cases}$$

**Answer**

$$f(x) = \begin{cases} 
\sin x - \cos x, & \text{if } x \neq 0 \\
-1 & \text{if } x = 0
\end{cases}$$

The given function $f$ is defined at all points of the real line.

Let $c$ be a real number.

**Case I:**

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If \( c \neq 0 \), then \( f(c) = \sin c - \cos c \)

\[
\lim_{x \to c} f(x) = \lim_{x \to c} (\sin x - \cos x) = \sin c - \cos c
\]

\[
\therefore \lim_{x \to c} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all points \( x \), such that \( x \neq 0 \)

Case II:

If \( c = 0 \), then \( f(0) = -1 \)

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1
\]

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1
\]

\[
\therefore \lim_{x \to 0} f(x) = \lim_{x \to 0} f(x) = f(0)
\]

Therefore, \( f \) is continuous at \( x = 0 \)

From the above observations, it can be concluded that \( f \) is continuous at every point of the real line.

Thus, \( f \) is a continuous function.

**Question 26:**

Find the values of \( k \) so that the function \( f \) is continuous at the indicated point.

\[
f(x) = \begin{cases} 
\frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\
3, & \text{if } x = \frac{\pi}{2}
\end{cases}
\]

at \( x = \frac{\pi}{2} \)

Answer

\[
f(x) = \begin{cases} 
\frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\
3, & \text{if } x = \frac{\pi}{2}
\end{cases}
\]

The given function \( f \) is

The given function \( f \) is continuous at \( x = \frac{\pi}{2} \), if \( f \) is defined at \( x = \frac{\pi}{2} \) and if the value of the \( f \)

\( x = \frac{\pi}{2} \) equals the limit of \( f \) at \( x = \frac{\pi}{2} \).
It is evident that $f$ is defined at $x = \frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = 3$

\[
\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x}
\]

Put $x = \frac{\pi}{2} + h$

Then, $x \to \frac{\pi}{2} \Rightarrow h \to 0$

\[
\therefore \lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x} = \lim_{h \to 0} \frac{k \cos \left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)}
\]

\[
= k \lim_{h \to 0} \frac{-\sin h}{-2h} = \frac{k}{2} \lim_{h \to 0} \frac{\sin h}{h} = \frac{k}{2} \cdot 1 = \frac{k}{2}
\]

\[
\therefore \lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)
\]

$\Rightarrow \frac{k}{2} = 3$

$\Rightarrow k = 6$

Therefore, the required value of $k$ is 6.

**Question 27:**

Find the values of $k$ so that the function $f$ is continuous at the indicated point.

\[f(x) = \begin{cases} 
  kx^2, & \text{if } x \leq 2 \\
  3, & \text{if } x > 2
\end{cases} \quad \text{at } x = 2\]

Answer

\[f(x) = \begin{cases} 
  kx^2, & \text{if } x \leq 2 \\
  3, & \text{if } x > 2
\end{cases}
\]

The given function is

The given function $f$ is continuous at $x = 2$, if $f$ is defined at $x = 2$ and if the value of $f$ at $x = 2$ equals the limit of $f$ at $x = 2$

It is evident that $f$ is defined at $x = 2$ and $f(2) = k(2)^2 = 4k$
\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} f(x) = f(2)
\]
\[
\Rightarrow \lim_{x \to 2} (kx^2) = \lim_{x \to 2} (3x) = 4k
\]
\[
\Rightarrow k \times 2^2 = 3 = 4k
\]
\[
\Rightarrow 4k = 3
\]
\[
\Rightarrow k = \frac{3}{4}
\]

Therefore, the required value of \(k\) is \(\frac{3}{4}\).

**Question 28:**

Find the values of \(k\) so that the function \(f\) is continuous at the indicated point.

\[f(x) = \begin{cases} 
  kx + 1, & \text{if } x \leq \pi \\
  \cos x, & \text{if } x > \pi 
\end{cases}\]

at \(x = \pi\)

**Answer**

\[f(x) = \begin{cases} 
  kx + 1, & \text{if } x \leq \pi \\
  \cos x, & \text{if } x > \pi 
\end{cases}\]

The given function is

The given function \(f\) is continuous at \(x = p\), if \(f\) is defined at \(x = p\) and if the value of \(f\) at \(x = p\) equals the limit of \(f\) at \(x = p\)

It is evident that \(f\) is defined at \(x = p\) and \(f(\pi) = k\pi + 1\)

\[
\lim_{x \to \pi} f(x) = \lim_{x \to \pi} f(x) = f(\pi)
\]
\[
\Rightarrow \lim_{x \to \pi} (kx + 1) = \lim_{x \to \pi} \cos x = k\pi + 1
\]
\[
\Rightarrow k\pi + 1 = \cos \pi = k\pi + 1
\]
\[
\Rightarrow k\pi + 1 = -1 = k\pi + 1
\]
\[
\Rightarrow k = -\frac{2}{\pi}
\]

Therefore, the required value of \(k\) is \(-\frac{2}{\pi}\).
Question 29:
Find the values of \( k \) so that the function \( f \) is continuous at the indicated point.
\[
f(x) = \begin{cases} 
  kx + 1, & \text{if } x \leq 5 \\
  3x - 5, & \text{if } x > 5 
\end{cases}
\] at \( x = 5 \)

Answer

\[ f(x) = \begin{cases} 
  kx + 1, & \text{if } x \leq 5 \\
  3x - 5, & \text{if } x > 5 
\end{cases} \]

The given function \( f \) is continuous at \( x = 5 \), if \( f \) is defined at \( x = 5 \) and if the value of \( f \) at \( x = 5 \) equals the limit of \( f \) at \( x = 5 \).

It is evident that \( f \) is defined at \( x = 5 \) and \( f(5) = k(5) + 1 = 5k + 1 \)

\[ \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5) \]

\[ \Rightarrow \lim_{x \to 5^-} (kx + 1) = \lim_{x \to 5^+} (3x - 5) = 5k + 1 \]

\[ \Rightarrow 5k + 1 = 5 \]

\[ \Rightarrow 5k = 4 \]

\[ \Rightarrow k = \frac{4}{5} \]

Therefore, the required value of \( k \) is \( \frac{4}{5} \).

Question 30:
Find the values of \( a \) and \( b \) such that the function defined by
\[
f(x) = \begin{cases} 
  5, & \text{if } x \leq 2 \\
  ax + b, & \text{if } 2 < x < 10 \\
  21, & \text{if } x \geq 10 
\end{cases}
\]
is a continuous function.
Answer

\[ f(x) = \begin{cases} 
5, & \text{if } x \leq 2 \\
ax + b, & \text{if } 2 < x < 10 \\
21, & \text{if } x \geq 10 
\end{cases} \]

The given function \( f \) is

It is evident that the given function \( f \) is defined at all points of the real line.

If \( f \) is a continuous function, then \( f \) is continuous at all real numbers.

In particular, \( f \) is continuous at \( x = 2 \) and \( x = 10 \)

Since \( f \) is continuous at \( x = 2 \), we obtain

\[ \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) \]

\[ \Rightarrow 5 = \lim_{x \to 2^-} (ax + b) = 5 \]

\[ \Rightarrow 2a + b = 5 \]

\[ \Rightarrow 2a + b = 5 \quad \ldots (1) \]

Since \( f \) is continuous at \( x = 10 \), we obtain

\[ \lim_{x \to 10^-} f(x) = \lim_{x \to 10^+} f(x) = f(10) \]

\[ \Rightarrow \lim_{x \to 10^-} (ax + b) = \lim_{x \to 10^+} (21) = 21 \]

\[ \Rightarrow 10a + b = 21 = 21 \]

\[ \Rightarrow 10a + b = 21 \quad \ldots (2) \]

On subtracting equation (1) from equation (2), we obtain

\[ 8a = 16 \]

\[ \Rightarrow a = 2 \]

By putting \( a = 2 \) in equation (1), we obtain

\[ 2 \times 2 + b = 5 \]

\[ \Rightarrow 4 + b = 5 \]
\[ b = 1 \]

Therefore, the values of \( a \) and \( b \) for which \( f \) is a continuous function are 2 and 1 respectively.

**Question 31:**
Show that the function defined by \( f(x) = \cos(x^2) \) is a continuous function.

**Answer**
The given function is \( f(x) = \cos(x^2) \)
This function \( f \) is defined for every real number and \( f \) can be written as the composition of two functions as,
\[
f = g \circ h, \quad \text{where} \quad g(x) = \cos x \quad \text{and} \quad h(x) = x^2
\]
\[
\therefore \left(g \circ h\right)(x) = g(h(x)) = g(x^2) = \cos(x^2) = f(x)
\]
It has to be first proved that \( g(x) = \cos x \) and \( h(x) = x^2 \) are continuous functions.
It is evident that \( g \) is defined for every real number.
Let \( c \) be a real number.
Then, \( g(c) = \cos c \)
Put \( x = c + h \)
If \( x \to c \), then \( h \to 0 \)
\[
\lim_{x \to c} g(x) = \lim_{x \to c} \cos x
\]
\[
= \lim_{h \to 0} \cos(c + h)
\]
\[
= \lim_{h \to 0} \left[\cos c \cos h - \sin c \sin h\right]
\]
\[
= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h
\]
\[
= \cos c \times 1 - \sin c \times 0
\]
\[
= \cos c
\]
\[
\therefore \lim_{x \to c} g(x) = g(c)
\]
Therefore, \( g(x) = \cos x \) is continuous function.
\[ h(x) = x^2 \]
Clearly, \( h \) is defined for every real number.

Let \( k \) be a real number, then \( h(k) = k^2 \)

\[
\lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2
\]
\[
\therefore \lim_{x \to k} h(x) = h(k)
\]

Therefore, \( h \) is a continuous function.

It is known that for real valued functions \( g \) and \( h \), such that \((g \circ h)\) is defined at \( c \), if \( g \) is continuous at \( c \) and if \( f \) is continuous at \( g(c) \), then \((f \circ g)\) is continuous at \( c \).

Therefore, \( f(x) = (g \circ h)(x) = \cos(x^2) \) is a continuous function.

**Question 32:**

Show that the function defined by \( f(x) = |\cos x| \) is a continuous function.

**Answer**

The given function is \( f(x) = |\cos x| \)

This function \( f \) is defined for every real number and \( f \) can be written as the composition of two functions as,

\[ f = g \circ h, \text{ where } g(x) = |x| \text{ and } h(x) = \cos x \]

\[ \therefore (g \circ h)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x) \]

It has to be first proved that \( g(x) = |x| \text{ and } h(x) = \cos x \) are continuous functions.

\[ g(x) = |x| \text{ can be written as } \]

\[ g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases} \]

Clearly, \( g \) is defined for all real numbers.

Let \( c \) be a real number.

**Case I:**

If \( c < 0 \), then \( g(c) = -c \) and \( \lim_{x \to c} g(x) = \lim_{x \to -c} (-x) = -c \)

\[ \therefore \lim_{x \to c} g(x) = g(c) \]
Therefore, \( g \) is continuous at all points \( x \), such that \( x < 0 \)

Case II:

If \( c > 0 \), then \( g(c) = c \) and \( \lim_{x \to c} g(x) = \lim_{x \to c} x = c \)

\[ \therefore \lim_{x \to c} g(x) = g(c) \]

Therefore, \( g \) is continuous at all points \( x \), such that \( x > 0 \)

Case III:

If \( c = 0 \), then \( g(c) = g(0) = 0 \)

\[ \lim_{x \to 0} g(x) = \lim_{x \to 0} (-x) = 0 \]

\[ \lim_{x \to 0} g(x) = \lim_{x \to 0} (x) = 0 \]

\[ \therefore \lim_{x \to 0} g(x) = \lim_{x \to 0} (x) = g(0) \]

Therefore, \( g \) is continuous at \( x = 0 \)

From the above three observations, it can be concluded that \( g \) is continuous at all points.

\( h(x) = \cos x \)

It is evident that \( h(x) = \cos x \) is defined for every real number.

Let \( c \) be a real number. Put \( x = c + h \)

If \( x \to c \), then \( h \to 0 \)

\( h(c) = \cos c \)

\[ \lim_{x \to c} h(x) = \lim_{x \to c} \cos x \]

\[ = \lim_{h \to 0} \cos(c + h) \]

\[ = \lim_{h \to 0} \left[ \cos c \cos h - \sin c \sin h \right] \]

\[ = \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h \]

\[ = \cos c \times 1 - \sin c \times 0 \]

\[ = \cos c \]

\[ \therefore \lim_{x \to c} h(x) = h(c) \]

Therefore, \( h(x) = \cos x \) is a continuous function.

It is known that for real valued functions \( g \) and \( h \), such that \( (g \circ h) \) is defined at \( c \), if \( g \) is continuous at \( c \) and if \( f \) is continuous at \( g(c) \), then \( (f \circ g) \) is continuous at \( c \).
Therefore, \( f(x) = (goh)(x) = g(h(x)) = g(\cos x) = |\cos x| \) is a continuous function.

**Question 33:**

Examine that \( \sin|x| \) is a continuous function.

**Answer**

Let \( f(x) = \sin|x| \)

This function \( f \) is defined for every real number and \( f \) can be written as the composition of two functions as,

\[
f = g \circ h, \quad \text{where } g(x) = |x| \text{ and } h(x) = \sin x
\]

\[
[\therefore (goh)(x) = g(h(x)) = g(\sin x) = |\sin x| = f(x)]
\]

It has to be proved first that \( g(x) = |x| \) and \( h(x) = \sin x \) are continuous functions.

\[
g(x) = |x| \text{ can be written as }
\]

\[
g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}
\]

Clearly, \( g \) is defined for all real numbers.

Let \( c \) be a real number.

**Case I:**

If \( c < 0 \), then \( g(c) = -c \) and \( \lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c \)

\[
\therefore \lim_{x \to c} g(x) = g(c)
\]

Therefore, \( g \) is continuous at all points \( x \), such that \( x < 0 \)

**Case II:**

If \( c > 0 \), then \( g(c) = c \) and \( \lim_{x \to c} g(x) = \lim_{x \to c} x = c \)

\[
\therefore \lim_{x \to c} g(x) = g(c)
\]

Therefore, \( g \) is continuous at all points \( x \), such that \( x > 0 \)

**Case III:**

If \( c = 0 \), then \( g(c) = g(0) = 0 \)
\[
\lim_{x \to 0} g(x) = \lim_{x \to 0} (-x) = 0
\]
\[
\lim_{x \to 0} g(x) = \lim_{x \to 0} (x) = 0
\]
\[\therefore \lim_{x \to 0} g(x) = \lim_{x \to 0} (x) = g(0)\]

Therefore, \(g\) is continuous at \(x = 0\).

From the above three observations, it can be concluded that \(g\) is continuous at all points.

\(h(x) = \sin x\)

It is evident that \(h(x) = \sin x\) is defined for every real number.

Let \(c\) be a real number. Put \(x = c + k\)

If \(x \to c\), then \(k \to 0\)

\(h(c) = \sin c\)

\[h(c) = \sin c\]

\[
\lim_{x \to c} h(x) = \lim_{x \to c} \sin x
\]

\[= \lim_{k \to 0} \sin (c + k)\]

\[= \lim_{k \to 0} \sin c \cos k + \cos c \sin k\]

\[= \lim_{k \to 0} (\sin c \cos k) + \lim_{k \to 0} (\cos c \sin k)\]

\[= \sin c \cos 0 + \cos c \sin 0\]

\[= \sin c + 0\]

\[= \sin c\]

\[\therefore \lim_{x \to c} h(x) = g(c)\]

Therefore, \(h\) is a continuous function.

It is known that for real valued functions \(g\) and \(h\), such that \((g \circ h)\) is defined at \(c\), if \(g\) is continuous at \(c\) and if \(f\) is continuous at \(g(c)\), then \((f \circ g)\) is continuous at \(c\).

Therefore, \(f(x) = (g \circ h)(x) = g(h(x)) = g(\sin x) = |\sin x|\) is a continuous function.

**Question 34:**

Find all the points of discontinuity of \(f\) defined by \(f(x) = |x| - |x + 1|\).

**Answer**

The given function is \(f(x) = |x| - |x + 1|\).
The two functions, \( g \) and \( h \), are defined as
\[
g(x) = |x| \quad \text{and} \quad h(x) = |x+1|
\]
Then, \( f = g - h \)
The continuity of \( g \) and \( h \) is examined first.

\[
g(x) = |x| \quad \text{can be written as}
\]
\[
g(x) = \begin{cases} 
-x, & \text{if } x < 0 \\
x, & \text{if } x \geq 0
\end{cases}
\]
Clearly, \( g \) is defined for all real numbers.
Let \( c \) be a real number.

**Case I:**
If \( c < 0 \), then \( g(c) = -c \) and \( \lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c \)
\[
\therefore \lim_{x \to c} g(x) = g(c)
\]
Therefore, \( g \) is continuous at all points \( x \), such that \( x < 0 \)

**Case II:**
If \( c > 0 \), then \( g(c) = c \) and \( \lim_{x \to c} g(x) = \lim_{x \to c} x = c \)
\[
\therefore \lim_{x \to c} g(x) = g(c)
\]
Therefore, \( g \) is continuous at all points \( x \), such that \( x > 0 \)

**Case III:**
If \( c = 0 \), then \( g(c) = g(0) = 0 \)
\[
\lim_{x \to 0} g(x) = \lim_{x \to 0} (-x) = 0 \\
\lim_{x \to 0} g(x) = \lim_{x \to 0} (x) = 0
\]
\[
\therefore \lim_{x \to 0} g(x) = \lim_{x \to 0} (x) = g(0)
\]
Therefore, \( g \) is continuous at \( x = 0 \)

From the above three observations, it can be concluded that \( g \) is continuous at all points.

\[
h(x) = |x+1| \quad \text{can be written as}
\]
\[
h(x) = \begin{cases} 
-(x+1), & \text{if } x < -1 \\
x+1, & \text{if } x \geq -1
\end{cases}
\]
Clearly, \( h \) is defined for every real number.
Let \( c \) be a real number.

Case I:
If \( c < -1 \), then \( h(c) = -(c+1) \) and \( \lim_{x \to c} h(x) = \lim_{x \to c} \left[ -(x+1) \right] = -(c+1) \)
\[ \therefore \lim_{x \to c} h(x) = h(c) \]
Therefore, \( h \) is continuous at all points \( x \), such that \( x < -1 \)

Case II:
If \( c > -1 \), then \( h(c) = c + 1 \) and \( \lim_{x \to c} h(x) = \lim_{x \to c} (x+1) = c + 1 \)
\[ \therefore \lim_{x \to c} h(x) = h(c) \]
Therefore, \( h \) is continuous at all points \( x \), such that \( x > -1 \)

Case III:
If \( c = -1 \), then \( h(c) = h(-1) = -1 + 1 = 0 \)
\[ \lim_{x \to -1} h(x) = \lim_{x \to -1} \left[ -(x+1) \right] = -(-1 + 1) = 0 \]
\[ \lim_{x \to -1} h(x) = \lim_{x \to -1} (x+1) = (-1 + 1) = 0 \]
\[ \therefore \lim_{x \to -1} h(x) = \lim_{h \to -1} h(x) = h(-1) \]
Therefore, \( h \) is continuous at \( x = -1 \)

From the above three observations, it can be concluded that \( h \) is continuous at all points of the real line.

\( g \) and \( h \) are continuous functions. Therefore, \( f = g - h \) is also a continuous function.

Therefore, \( f \) has no point of discontinuity.
Exercise 5.2

Question 1:
Differentiate the functions with respect to \( x \).

\[ \sin (x^2 + 5) \]

Answer

Let \( f(x) = \sin (x^2 + 5) \), \( u(x) = x^2 + 5 \), and \( v(t) = \sin t \)

Then, \( f(x) = v(u(x)) = \sin (x^2 + 5) = \tan (x^2 + 5) = f(x) \)

Thus, \( f \) is a composite of two functions.

Put \( t = u(x) = x^2 + 5 \)

Then, we obtain

\[
\frac{df}{dt} = \frac{d}{dt} (\sin t) = \cos t = \cos (x^2 + 5)
\]

\[
\frac{dt}{dx} = \frac{d}{dx} (x^2 + 5) = 2x + \frac{d}{dx} (5) = 2x + 0 = 2x
\]

Therefore, by chain rule, \( \frac{df}{dx} = \frac{df}{dt} \frac{dt}{dx} = \cos (x^2 + 5) \times 2x = 2x \cos (x^2 + 5) \)

Alternate method

\[
\frac{d}{dx} \left[ \sin (x^2 + 5) \right] = \cos (x^2 + 5) \cdot \frac{d}{dx} (x^2 + 5)
\]

\[
= \cos (x^2 + 5) \cdot \left[ 2x + \frac{d}{dx} (5) \right]
\]

\[
= 2x \cos (x^2 + 5)
\]

Question 2:
Differentiate the functions with respect to \( x \).

\[ \cos (\sin x) \]
Answer

Let \( f(x) = \cos(\sin x), u(x) = \sin x, \) and \( v(t) = \cos t \)

Then, \( (v \circ u)(x) = v(u(x)) = v(\sin x) = \cos(\sin x) = f(x) \)

Thus, \( f \) is a composite function of two functions.

Put \( t = u(x) = \sin x \)

:. \( \frac{dv}{dt} = \frac{d}{dt}[\cos t] = -\sin t = -\sin(\sin x) \)

\[ \frac{dt}{dx} = \frac{d}{dx}(\sin x) = \cos x \]

\[ \frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x) \]

By chain rule, \( \frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x) \)

Alternate method

\[ \frac{d}{dx}\left[\cos(\sin x)\right] = -\sin(\sin x) \cdot \frac{d}{dx}(\sin x) = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x) \]

Question 3:
Differentiate the functions with respect to \( x \).

\( \sin(ax+b) \)

Answer

Let \( f(x) = \sin(ax+b), u(x) = ax+b, \) and \( v(t) = \sin t \)

Then, \( (v \circ u)(x) = v(u(x)) = v(ax+b) = \sin(ax+b) = f(x) \)

Thus, \( f \) is a composite function of two functions, \( u \) and \( v \).

Put \( t = u(x) = ax+b \)

Therefore,

\[ \frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax+b) \]

\[ \frac{dt}{dx} = \frac{d}{dx}(ax+b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a \]

Hence, by chain rule, we obtain

\[ \frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax+b) \cdot a = a \cos(ax+b) \]

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Alternate method

\[
\frac{d}{dx} \left[ \sin (ax + b) \right] = \cos (ax + b) \cdot \frac{d}{dx} (ax + b) \\
= \cos (ax + b) \cdot \left[ \frac{d}{dx} (ax) + \frac{d}{dx} (b) \right] \\
= \cos (ax + b) \cdot (a + 0) \\
= a \cos (ax + b)
\]

Question 4:
Differentiate the functions with respect to \( x \).

\[ \sec \left( \tan \left( \sqrt{x} \right) \right) \]

Answer

Let \( f(x) = \sec \left( \tan \sqrt{x} \right) \), \( u(x) = \sqrt{x} \), \( v(t) = \tan t \), and \( w(s) = \sec s \)

Then, \( (w(v(u)))'(x) = w'[v(u(x))] = w'\left( \sqrt{x} \right) = w(\tan \sqrt{x}) = \sec (\tan \sqrt{x}) = f(x) \)

Thus, \( f \) is a composite function of three functions, \( u, v, \) and \( w \).

Put \( s = v(t) = \tan t \) and \( t = u(x) = \sqrt{x} \)

Then, \( \frac{dw}{ds} = \frac{d}{ds} (\sec s) = \sec s \tan s = \sec (\tan t) \cdot \tan (\tan t) \quad [s = \tan t] \)

\[ = \sec \left( \tan \sqrt{x} \right) \cdot \tan \left( \tan \sqrt{x} \right) \quad [t = \sqrt{x}] \]

\[ \frac{ds}{dt} = \frac{d}{dt} (\tan t) = \sec^2 t = \sec^2 \sqrt{x} \]

\[ \frac{dt}{dx} = \frac{d}{dx} \left( \sqrt{x} \right) = \frac{d}{dx} \left( x^{\frac{1}{2}} \right) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \]

Hence, by chain rule, we obtain
\[ \frac{dt}{dx} = \frac{d\tan\sqrt{x}}{dx} \cdot \tan\left(\tan\sqrt{x}\right) \times \sec^2\sqrt{x} \times \frac{1}{2\sqrt{x}} \]

\[ = \frac{1}{2\sqrt{x}} \sec^2\sqrt{x} \sec\left(\tan\sqrt{x}\right) \tan\left(\tan\sqrt{x}\right) \]

\[ = \frac{\sec^2\sqrt{x} \sec\left(\tan\sqrt{x}\right) \tan\left(\tan\sqrt{x}\right)}{2\sqrt{x}} \]

**Alternate method**

\[ \frac{d}{dx}\left[ \sec\left(\tan\sqrt{x}\right) \right] = \sec\left(\tan\sqrt{x}\right) \cdot \tan\left(\tan\sqrt{x}\right) \cdot \frac{d}{dx}\left(\tan\sqrt{x}\right) \]

\[ = \sec\left(\tan\sqrt{x}\right) \cdot \tan\left(\tan\sqrt{x}\right) \cdot \sec^2\left(\sqrt{x}\right) \cdot \frac{d}{dx}\left(\sqrt{x}\right) \]

\[ = \sec\left(\tan\sqrt{x}\right) \cdot \tan\left(\tan\sqrt{x}\right) \cdot \sec^2\left(\sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}} \]

\[ = \frac{\sec\left(\tan\sqrt{x}\right) \cdot \tan\left(\tan\sqrt{x}\right) \cdot \sec^2\left(\sqrt{x}\right)}{2\sqrt{x}} \]

**Question 5:**

Differentiate the functions with respect to \( x \).

\[ \frac{\sin{(ax+b)}}{\cos{(cx+d)}} \]

**Answer**

\[ f(x) = \frac{\sin{(ax+b)}}{\cos{(cx+d)}} = \frac{g(x)}{h(x)} \]

The given function is

\[ h(x) = \cos{(cx+d)} \]

\[ g(x) = \sin{(ax+b)} \]

\[ \therefore f' = \frac{g'h - gh'}{h^2} \]

Consider \( g(x) = \sin{(ax+b)} \)

Let \( u(x) = ax + b \), \( v(t) = \sin t \)

Then, \( v(u(x)) = v(ax+b) = \sin(ax+b) = g(x) \)
∴ $g$ is a composite function of two functions, $u$ and $v$.

Put $t = u(x) = ax + b$

\[
\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax + b)
\]

\[
\frac{dt}{dx} = \frac{d}{dx}(ax + b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a
\]

Therefore, by chain rule, we obtain

\[
g' = \frac{dg}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax + b) \cdot a = a \cos(ax + b)
\]

Consider $h(x) = \cos(cx + d)$

Let $p(x) = cx + d$, $q(y) = \cos y$

Then, $(q \circ p)(x) = q(p(x)) = q(cx + d) = \cos(cx + d) = h(x)$

∴ $h$ is a composite function of two functions, $p$ and $q$.

Put $y = p(x) = cx + d$

\[
\frac{dq}{dy} = \frac{d}{dy}(\sin y) = -\sin y = -\sin(cx + d)
\]

\[
\frac{dy}{dx} = \frac{d}{dx}(cx + d) = \frac{d}{dx}(cx) + \frac{d}{dx}(d) = c
\]

Therefore, by chain rule, we obtain

\[
h' = \frac{dh}{dx} = \frac{dq}{dy} \cdot \frac{dy}{dx} = -\sin(cx + d) \times c = -c \sin(cx + d)
\]
\[ f'' = \frac{a \cos(ax + b) \cdot \cos(cx + d) - \sin(ax + b) \cdot \sin(cx + d)}{[\cos(cx + d)]^2} \]
\[ = \frac{a \cos(ax + b)}{\cos(cx + d)} + c \sin(ax + b) \cdot \frac{\sin(cx + d)}{\cos(cx + d)} \cdot \tan(cx + d) \cdot \sec(cx + d) \]

Question 6:
Differentiate the functions with respect to \( x \).
\[ \cos(x^3) \cdot \sin^2(x^5) \]
Answer
The given function is \( \cos(x^3) \cdot \sin^2(x^5) \).
\[ \frac{d}{dx} \left[ \cos(x^3) \cdot \sin^2(x^5) \right] = \sin^2(x^5) \cdot \frac{d}{dx} \left( \cos(x^3) \right) + \cos(x^3) \cdot \frac{d}{dx} \left[ \sin^2(x^5) \right] \]
\[ = \sin^2(x^5) \cdot (-\sin(x^3)) \cdot \frac{d}{dx} (x^3) + \cos(x^3) \cdot 2 \sin(x^5) \cdot \frac{d}{dx} \left[ \sin(x^5) \right] \]
\[ = -\sin(x^3) \cdot \sin^2(x^5) \cdot 3x^2 + 2 \sin(x^5) \cdot \cos(x^5) \cdot \cos(x^3) \cdot 5x^4 \]
\[ = 10x^4 \cdot \sin(x^5) \cdot \cos(x^3) - 3x^2 \cdot \sin(x^3) \cdot \sin^2(x^5) \]

Question 7:
Differentiate the functions with respect to \( x \).
\[ 2 \sqrt{\cot(x^2)} \]
Answer
\[
\frac{d}{dx} \left[ 2\sqrt{\cot(x^2)} \right] \\
= 2 \cdot \frac{1}{2\sqrt{\cot(x^2)}} \times \frac{d}{dx} \left[ \cot(x^2) \right] \\
= \frac{\sin(x^2)}{\cos(x^2)} \times \frac{1}{\sin^2(x^2)} \times \frac{d}{dx} (x^2) \\
= \frac{-2x}{\sqrt{\cos(x^2) \sin(x^2) \sin x^2}} \\
= \frac{-2\sqrt{2}x}{\sqrt{2} \sin x^2 \cos x^2 \sin x^2} \\
= \frac{-2\sqrt{2}x}{\sin x^2 \sin 2x^2}
\]

**Question 8:**
Differentiate the functions with respect to \( x \).

\[\cos(\sqrt{x})\]

**Answer**

Let \( f(x) = \cos(\sqrt{x}) \)

Also, let \( u(x) = \sqrt{x} \)

And, \( v(t) = \cos t \)

Then, \((v \circ u)(x) = v(u(x))\)

\[= v(\sqrt{x})\]

\[= \cos \sqrt{x}\]

\[= f(x)\]

Clearly, \( f \) is a composite function of two functions, \( u \) and \( v \), such that

\[t = u(x) = \sqrt{x}\]

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Then, \[ \frac{dt}{dx} = \frac{d}{dx} \left( \sqrt{x} \right) = \frac{d}{dx} \left( x^{\frac{1}{2}} \right) = \frac{1}{2} x^{-\frac{1}{2}} \]

\[ = \frac{1}{2\sqrt{x}} \]

And, \[ \frac{dy}{dt} = \frac{d}{dt} (\cos t) = -\sin t \]

\[ = -\sin \left( \sqrt{x} \right) \]

By using chain rule, we obtain

\[ \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \]

\[ = -\sin \left( \sqrt{x} \right) \cdot \frac{1}{2\sqrt{x}} \]

\[ = -\frac{1}{2\sqrt{x}} \sin \left( \sqrt{x} \right) \]

\[ = -\frac{\sin \left( \sqrt{x} \right)}{2\sqrt{x}} \]

Alternate method

\[ \frac{d}{dx} \left[ \cos \left( \sqrt{x} \right) \right] = -\sin \left( \sqrt{x} \right) \cdot \frac{d}{dx} \left( \sqrt{x} \right) \]

\[ = -\sin \left( \sqrt{x} \right) \cdot \frac{d}{dx} \left( x^{\frac{1}{2}} \right) \]

\[ = -\sin \sqrt{x} \cdot \frac{1}{2} x^{-\frac{1}{2}} \]

\[ = -\frac{\sin \sqrt{x}}{2\sqrt{x}} \]

Question 9:
Prove that the function \( f \) given by

\[ f(x) = |x-1|, \ x \in \mathbb{R} \]

is not differentiable at \( x = 1 \).

Answer

The given function is \( f(x) = |x-1|, \ x \in \mathbb{R} \)
It is known that a function \( f \) is differentiable at a point \( x = c \) in its domain if both
\[
\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \text{ and } \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}
\]
are finite and equal.

To check the differentiability of the given function at \( x = 1 \),
consider the left hand limit of \( f \) at \( x = 1 \)
\[
\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{|1+h-1|-|1-1|}{h} = \lim_{h \to 0^-} \frac{|h|-0}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1
\]

Consider the right hand limit of \( f \) at \( x = 1 \)
\[
\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{|1+h-1|-|1-1|}{h} = \lim_{h \to 0^+} \frac{|h|-0}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1
\]

Since the left and right hand limits of \( f \) at \( x = 1 \) are not equal, \( f \) is not differentiable at \( x = 1 \)

**Question 10:**

Prove that the greatest integer function defined by \( f(x) = [x], 0 < x < 3 \) is not differentiable at \( x = 1 \) and \( x = 2 \).

**Answer**

The given function \( f \) is \( f(x) = [x], 0 < x < 3 \)

It is known that a function \( f \) is differentiable at a point \( x = c \) in its domain if both
\[
\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \text{ and } \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}
\]
are finite and equal.

To check the differentiability of the given function at \( x = 1 \), consider the left hand limit of \( f \) at \( x = 1 \)
Since the left and right hand limits of \( f \) at \( x = 1 \) are not equal, \( f \) is not differentiable at \( x = 1 \).

To check the differentiability of the given function at \( x = 2 \), consider the left hand limit of \( f \) at \( x = 2 \):

\[
\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{[2+h] - [2]}{h} = \lim_{h \to 0^-} \frac{1-2}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty
\]

Consider the right hand limit of \( f \) at \( x = 1 \):

\[
\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{[2+h] - [2]}{h} = \lim_{h \to 0^+} \frac{2-2}{h} = \lim_{h \to 0^+} \frac{0}{h} = 0
\]

Since the left and right hand limits of \( f \) at \( x = 2 \) are not equal, \( f \) is not differentiable at \( x = 2 \).
Exercise 5.3

Question 1:

\[ \frac{dy}{dx} \]

Find \( \frac{dx}{dx} \):

\[ 2x + 3y = \sin x \]

Answer

The given relationship is \( 2x + 3y = \sin x \)

Differentiating this relationship with respect to \( x \), we obtain

\[ \frac{d}{dx}(2x + 3y) = \frac{d}{dx}(\sin x) \]

\[ \Rightarrow \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \cos x \]

\[ \Rightarrow 2 + 3 \frac{dy}{dx} = \cos x \]

\[ \Rightarrow 3 \frac{dy}{dx} = \cos x - 2 \]

\[ \therefore \frac{dy}{dx} = \frac{\cos x - 2}{3} \]

Question 2:

\[ \frac{dy}{dx} \]

Find \( \frac{dx}{dx} \):

\[ 2x + 3y = \sin y \]

Answer

The given relationship is \( 2x + 3y = \sin y \)

Differentiating this relationship with respect to \( x \), we obtain

\[ \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(\sin y) \]
\[ \Rightarrow 2 + 3 \frac{dy}{dx} = \cos y \frac{dy}{dx} \]  
\[ \Rightarrow 2 = (\cos y - 3) \frac{dy}{dx} \]  
\[ \Rightarrow \frac{dy}{dx} = \frac{2}{\cos y - 3} \]

**Question 3:**

\[ \frac{dy}{dx} \]

Find \( \frac{dy}{dx} \):

\[ ax + by^2 = \cos y \]

**Answer**

The given relationship is \( ax + by^2 = \cos y \)

Differentiating this relationship with respect to \( x \), we obtain

\[ \frac{d}{dx}(ax) + \frac{d}{dx}(by^2) = \frac{d}{dx}(\cos y) \]

\[ \Rightarrow a + b \frac{d}{dx}(y^2) = \frac{d}{dx}(\cos y) \]

... (1)

\[ \frac{d}{dx}(y^2) = 2y \frac{dy}{dx} \text{ and } \frac{d}{dx}(\cos y) = -\sin y \frac{dy}{dx} \]

... (2)

Using chain rule, we obtain

From (1) and (2), we obtain

\[ a + b \times 2y \frac{dy}{dx} = -\sin y \frac{dy}{dx} \]

\[ \Rightarrow (2by + \sin y) \frac{dy}{dx} = -a \]

\[ \Rightarrow \frac{dy}{dx} = \frac{-a}{2by + \sin y} \]

**Question 4:**

\[ \frac{dy}{dx} \]

Find \( \frac{dy}{dx} \):

\[ xy + y^2 = \tan x + y \]
Answer

The given relationship is \( xy + y^2 = \tan x + y \)

Differentiating this relationship with respect to \( x \), we obtain

\[
\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(\tan x + y)
\]

\[
\Rightarrow \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x) + \frac{dy}{dx}
\]

\[
\Rightarrow \left[y \cdot \frac{d}{dx}(x) + x \cdot \frac{dy}{dx} \right] + 2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}
\]

\[
\Rightarrow y \cdot 1 + x \frac{dy}{dx} + 2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}
\]

\[
\Rightarrow (x + 2y - 1) \frac{dy}{dx} = \sec^2 x - y
\]

\[
\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1}
\]

Question 5:

\[
\frac{dy}{dx}
\]

Find \( \frac{dx}{dy} \):

\( x^2 + xy + y^2 = 100 \)

Answer

The given relationship is \( x^2 + xy + y^2 = 100 \)

Differentiating this relationship with respect to \( x \), we obtain

\[
\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(100)
\]

\[
\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0
\] [Derivative of constant function is 0]
Question 6:

\[
\frac{dy}{dx}
\]

Find \(\frac{dy}{dx}\):

\[x^3 + x^2 y + xy^2 + y^3 = 81\]

Answer

The given relationship is \(x^3 + x^2 y + xy^2 + y^3 = 81\)

Differentiating this relationship with respect to \(x\), we obtain

\[
\frac{d}{dx} \left( x^3 + x^2 y + xy^2 + y^3 \right) = \frac{d}{dx} (81)
\]

\[
\Rightarrow \frac{d}{dx} (x^3) + \frac{d}{dx} (x^2 y) + \frac{d}{dx} (xy^2) + \frac{d}{dx} (y^3) = 0
\]

\[
\Rightarrow 3x^2 + \left[ y \cdot \frac{d}{dx} (x^3) + x^3 \cdot \frac{dy}{dx} \right] + \left[ y^2 \cdot \frac{d}{dx} (x) + x \cdot \frac{d}{dx} (y^2) \right] + 3y^2 \frac{dy}{dx} = 0
\]

\[
\Rightarrow 3x^2 + \left[ y \cdot 2x + x^2 \frac{dy}{dx} \right] + \left[ y^2 \cdot 1 + x \cdot 2y \cdot \frac{dy}{dx} \right] + 3y^3 \frac{dy}{dx} = 0
\]

\[
\Rightarrow \left( x^2 + 2xy + y^3 \right) \frac{dy}{dx} + \left( 3x^2 + 2xy + y^3 \right) = 0
\]

\[
\therefore \frac{dy}{dx} = -\frac{\left( 3x^2 + 2xy + y^3 \right)}{\left( x^2 + 2xy + y^3 \right)}
\]

Question 7:

\[
\frac{dy}{dx}
\]

Find \(\frac{dy}{dx}\):

\[\sin^2 y + \cos xy = \pi\]
Answer

The given relationship is \( \sin^2 y + \cos xy = \pi \)

Differentiating this relationship with respect to \( x \), we obtain

\[
\frac{d}{dx} (\sin^2 y + \cos xy) = \frac{d}{dx} (\pi)
\]

\[
\Rightarrow \frac{d}{dx} (\sin^2 y) + \frac{d}{dx} (\cos xy) = 0
\]

...(1)

Using chain rule, we obtain

\[
\frac{d}{dx} (\sin^2 y) = 2 \sin y \frac{d}{dx} (\sin y) = 2 \sin y \cos y \frac{dy}{dx}
\]

...(2)

\[
\frac{d}{dx} (\cos xy) = -\sin xy \frac{d}{dx} (xy) = -\sin xy \left[ \frac{d}{dx} (x) + x \frac{dy}{dx} \right]
\]

\[
= -\sin xy \left[ 1 + x \frac{dy}{dx} \right] = -x \sin xy - x \sin xy \frac{dy}{dx}
\]

...(3)

From (1), (2), and (3), we obtain

\[
2 \sin y \cos y \frac{dy}{dx} - y \sin xy - x \sin xy \frac{dy}{dx} = 0
\]

\[
\Rightarrow (2 \sin y \cos y - x \sin xy) \frac{dy}{dx} = y \sin xy
\]

\[
\Rightarrow (\sin 2y - x \sin xy) \frac{dy}{dx} = y \sin xy
\]

\[
\therefore \frac{dy}{dx} = \frac{y \sin xy}{\sin 2y - x \sin xy}
\]

Question 8:

Find \( \frac{dy}{dx} \):

\( \sin^2 x + \cos^2 y = 1 \)

Answer

The given relationship is \( \sin^2 x + \cos^2 y = 1 \)

Differentiating this relationship with respect to \( x \), we obtain
\[ \frac{d}{dx} \left( \sin^2 x + \cos^2 y \right) = \frac{d}{dx} (1) \]
\[ \Rightarrow \frac{d}{dx} \left( \sin^2 x \right) + \frac{d}{dx} \left( \cos^2 y \right) = 0 \]
\[ \Rightarrow 2 \sin x \cdot \frac{d}{dx} (\sin x) + 2 \cos y \cdot \frac{d}{dx} (\cos y) = 0 \]
\[ \Rightarrow 2 \sin x \cos x + 2 \cos y (-\sin y) \cdot \frac{dy}{dx} = 0 \]
\[ \Rightarrow \sin 2x - \sin 2y \frac{dy}{dx} = 0 \]
\[ \therefore \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y} \]

**Question 9:**

\[ \frac{dy}{dx} \]

Find \( \frac{dy}{dx} \):

\[ y = \sin^{-1} \left( \frac{2x}{1 + x^2} \right) \]

**Answer**

\[ y = \sin^{-1} \left( \frac{2x}{1 + x^2} \right) \]

The given relationship is

\[ y = \sin^{-1} \left( \frac{2x}{1 + x^2} \right) \]

\[ \Rightarrow \sin y = \frac{2x}{1 + x^2} \]

Differentiating this relationship with respect to \( x \), we obtain

\[ \frac{d}{dx} \left( \sin y \right) = \frac{d}{dx} \left( \frac{2x}{1 + x^2} \right) \]

\[ \Rightarrow \cos y \frac{dy}{dx} = \frac{d}{dx} \left( \frac{2x}{1 + x^2} \right) \]

\[ \Rightarrow \cos y \frac{dy}{dx} = \frac{2x}{(1 + x^2)^2} \]

\[ \Rightarrow \frac{2x}{1 + x^2} \]

The function, \( \frac{2x}{1 + x^2} \), is of the form of \( \frac{u}{v} \).

Therefore, by quotient rule, we obtain
\[
\frac{d}{dx} \left( \frac{2x}{1 + x^2} \right) = \frac{(1 + x^2) \cdot \frac{d}{dx}(2x) - 2x \cdot \frac{d}{dx}(1 + x^2)}{(1 + x^2)^2} \\
= \frac{(1 + x^2) \cdot 2 - 2x \cdot [0 + 2x]}{(1 + x^2)^2} = \frac{2 + 2x^2 - 4x^2}{(1 + x^2)^2} = \frac{2(1 - x^2)}{(1 + x^2)^2} \quad \text{...(2)}
\]

Also,
\[
\sin y = \frac{2x}{1 + x^2}
\]

\Rightarrow \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \left( \frac{2x}{1 + x^2} \right)^2} = \sqrt{\frac{(1 + x^2)^2 - 4x^2}{(1 + x^2)^2}}
\]
\[
= \sqrt{\frac{(1 - x^2)^2}{(1 + x^2)^2}} = \frac{1 - x^2}{1 + x^2} \quad \text{...(3)}
\]

From (1), (2), and (3), we obtain
\[
\frac{1 - x^2}{1 + x^2} \times \frac{dy}{dx} = \frac{2(1 - x^2)}{(1 + x^2)^2}
\]

\Rightarrow \frac{dy}{dx} = \frac{2}{1 + x^2}

**Question 10:**

\[
\frac{dy}{dx} \\
\text{Find } \frac{dx}{dy}:
\]
\[
y = \tan^{-1} \left( \frac{3x - x^3}{1 - 3x^2} \right), \quad -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}
\]

Answer

\[
y = \tan^{-1} \left( \frac{3x - x^3}{1 - 3x^2} \right)
\]
The given relationship is
It is known that,
Comparing equations (1) and (2), we obtain

\[ x = \tan \frac{y}{3} \]

Differentiating this relationship with respect to \( x \), we obtain

\[
\frac{d}{dx} \left( x \right) = \frac{d}{dx} \left( \tan \frac{y}{3} \right)
\]

\[ \Rightarrow 1 = \sec^2 \frac{y}{3} \cdot \frac{dy}{3} \]

\[ \Rightarrow 1 = \sec^2 \frac{y}{3} \cdot \frac{1}{3} \cdot \frac{dy}{dx} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{3}{\sec^2 \frac{y}{3}} \cdot \frac{1}{1 + \tan^2 \frac{y}{3}} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{3}{1 + x^2} \]

**Question 11:**

Find \( \frac{dx}{dy} \):

\[ y = \cos^{-1}\left( \frac{1 - x^2}{1 + x^2} \right), 0 < x < 1 \]

**Answer**

The given relationship is,
\[ y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \]

\[ \Rightarrow \cos y = \frac{1-x^2}{1+x^2} \]

\[ \frac{1-\tan^2\frac{y}{2}}{2} = \frac{1-x^2}{1+x^2} \]

On comparing L.H.S. and R.H.S. of the above relationship, we obtain

\[ \tan \frac{y}{2} = x \]

Differentiating this relationship with respect to \( x \), we obtain

\[ \sec^2 \frac{y}{2} \frac{d}{dx} \left( \frac{y}{2} \right) = \frac{d}{dx} (x) \]

\[ \Rightarrow \sec^2 \frac{y}{2} \cdot \frac{1}{2} \frac{dy}{dx} = 1 \]

\[ \Rightarrow \frac{dy}{dx} = \frac{2}{\sec^2 \frac{y}{2}} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{2}{1 + \tan^2 \frac{y}{2}} \]

\[ \therefore \frac{dy}{dx} = \frac{1}{1+x^2} \]

Question 12:

Find \( \frac{dy}{dx} \):

\[ y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), \quad 0 < x < 1 \]

Answer

\[ y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) \]

The given relationship is
Differentiating this relationship with respect to \( x \), we obtain

\[
\frac{d}{dx} \left( \sin y \right) = \frac{d}{dx} \left( \frac{1-x^2}{1+x^2} \right)
\]

... (1)

Using chain rule, we obtain

\[
\frac{d}{dx} \left( \sin y \right) = \cos y \cdot \frac{dy}{dx}
\]

\[
\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}
\]

\[
= \sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}} = \sqrt{\frac{4x^2}{(1+x^2)^2}} = \frac{2x}{1+x^2}
\]

\[
\therefore \frac{d}{dx} \left( \sin y \right) = \frac{2x}{1+x^2} \cdot \frac{dy}{dx} \quad \text{... (2)}
\]

\[
\frac{d}{dx} \left( \frac{1-x^2}{1+x^2} \right) = \frac{(1+x^2) \cdot (1-x^2)' - (1-x^2) \cdot (1+x^2)'}{(1+x^2)^2}
\]

[Using quotient rule]

\[
= \frac{(1+x^2)(-2x)-(1-x^2) \cdot (2x)}{(1+x^2)^2}
\]

\[
= \frac{-2x - 2x^3 - 2x + 2x^3}{(1+x^2)^2}
\]

\[
= \frac{-4x}{(1+x^2)^2} \quad \text{... (3)}
\]

From (1), (2), and (3), we obtain
\[
\frac{2x}{1+x^2} \frac{dy}{dx} = \frac{-4x}{(1+x^2)^2}
\]

\[\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^3}\]

**Alternate method**

\[y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)\]

\[\Rightarrow \sin y = \frac{1-x^2}{1+x^2}\]

\[\Rightarrow (1+x^2)\sin y = 1 - x^2\]

\[\Rightarrow (1 + \sin y)x^2 = 1 - \sin y\]

\[\Rightarrow x^2 = \frac{1 - \sin y}{1 + \sin y}\]

\[\Rightarrow x^2 = \left(\frac{\cos \frac{y}{2} - \sin \frac{y}{2}}{\cos \frac{y}{2} + \sin \frac{y}{2}}\right)^2\]

\[\Rightarrow x = \frac{\cos \frac{y}{2} - \sin \frac{y}{2}}{\cos \frac{y}{2} + \sin \frac{y}{2}}\]

\[\Rightarrow x = \frac{1 - \tan \frac{y}{2}}{1 + \tan \frac{y}{2}}\]

\[\Rightarrow x = \tan\left(\frac{\pi}{4} - \frac{y}{2}\right)\]

Differentiating this relationship with respect to \(x\), we obtain
Question 13:

\[
\frac{dy}{dx}
\]

Find \( \frac{dx}{dy} \):

\[y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), \quad -1 < x < 1\]

Answer

The given relationship is

\[y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)\]

Differentiating this relationship with respect to \( x \), we obtain

\[\frac{d}{dx}(\cos y) = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)\]

\[\Rightarrow -\sin y \cdot \frac{dy}{dx} = \frac{(1+x^2) \cdot \frac{d}{dx}(2x) - 2x \cdot \frac{d}{dx}(1+x^2)}{(1+x^2)^2}\]
\[ \Rightarrow -\sqrt{1-\cos^2 y} \frac{dy}{dx} = \frac{(1+x^2)^2 - 2x^2}{(1+x^2)^2} \]

\[ \Rightarrow \sqrt{1-\left(\frac{2x}{1+x^2}\right)^2} \frac{dy}{dx} = -\frac{2(1-x^2)}{(1+x^2)^2} \]

\[ \Rightarrow \frac{(1+x^2)^2 - 4x^2}{(1+x^2)^2} \frac{dy}{dx} = -\frac{2(1-x^2)}{(1+x^2)^2} \]

\[ \Rightarrow \frac{(1-x^2)^2}{(1+x^2)^2} \frac{dy}{dx} = -\frac{2(1-x^2)}{(1+x^2)^2} \]

\[ \Rightarrow \frac{1-x^2}{1+x^2} \cdot \frac{dy}{dx} = -\frac{2(1-x^2)}{(1+x^2)^2} \]

\[ \Rightarrow \frac{dy}{dx} = -\frac{2}{1+x^2} \]

**Question 14:**

Find \( \frac{dy}{dx} : \)

\[ y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), \quad -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \]

**Answer**

The relationship is

\[ y = \sin^{-1}\left(2x\sqrt{1-x^2}\right) \]

\[ y = \sin^{-1}\left(2x\sqrt{1-x^2}\right) \]

\[ \Rightarrow \sin y = 2x\sqrt{1-x^2} \]

Differentiating this relationship with respect to \( x \), we obtain
Question 15:

\[ \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}} \]

Find \( \frac{dy}{dx} \):

\[ y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}} \]

Answer

The given relationship is

\[ y = \sec^{-1}\left(\frac{1}{2x^2-1}\right) \]
\[ \Rightarrow \sec y = \frac{1}{2x^2 - 1} \]
\[ \Rightarrow \cos y = 2x^2 - 1 \]
\[ \Rightarrow 2x^2 = 1 + \cos y \]
\[ \Rightarrow 2x^2 = 2 \cos^2 \frac{y}{2} \]
\[ \Rightarrow x = \cos \frac{y}{2} \]

Differentiating this relationship with respect to \( x \), we obtain

\[ \frac{d}{dx} \left( x \right) = \frac{d}{dx} \left( \cos \frac{y}{2} \right) \]
\[ \Rightarrow 1 = -\sin \frac{y}{2} \cdot \frac{d}{dx} \left( \frac{y}{2} \right) \]
\[ \Rightarrow \frac{-1}{\sin \frac{y}{2}} = \frac{1}{\frac{y}{2}} \cdot \frac{dy}{dx} \]
\[ \Rightarrow \frac{dy}{dx} = \frac{-2}{\sin \frac{y}{2}} \cdot \frac{-2}{\sqrt{1 - \cos^2 \frac{y}{2}}} \]
\[ \Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - x^2}} \]
Question 1:
Differentiate the following w.r.t. $x$:

\[ \frac{e^x}{\sin x} \]

Answer

Let \( y = \frac{e^x}{\sin x} \)

By using the quotient rule, we obtain

\[ \frac{dy}{dx} = \frac{\sin x \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(\sin x)}{\sin^2 x} \]

\[ = \frac{\sin x (e^x) - e^x \cdot (\cos x)}{\sin^2 x} \]

\[ = \frac{e^x (\sin x - \cos x)}{\sin^2 x}, x \neq n\pi, n \in \mathbb{Z} \]

Question 2:
Differentiate the following w.r.t. $x$:

\[ e^{\sin^{-1} x} \]

Answer

Let \( y = e^{\sin^{-1} x} \)

By using the chain rule, we obtain

\[ \frac{dy}{dx} = e^{\sin^{-1} x} \cdot \frac{d}{dx}(\sin^{-1} x) \]

\[ = e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1 - x^2}} \]
Question 2:
Show that the function given by \( f(x) = e^{2x} \) is strictly increasing on \( \mathbb{R} \).

Answer

Let \( x_1 \) and \( x_2 \) be any two numbers in \( \mathbb{R} \).

Then, we have:

\[ x_1 < x_2 \implies 2x_1 < 2x_2 \implies e^{2x_1} < e^{2x_2} = f(x_1) < f(x_2) \]

Hence, \( f \) is strictly increasing on \( \mathbb{R} \).

Question 3:
Differentiate the following w.r.t. \( x \):

\( e^{x^3} \)

Answer

Let \( y = e^{x^3} \)

By using the chain rule, we obtain

\[ \frac{dy}{dx} = \frac{d}{dx} \left( e^{x^3} \right) = e^{x^3} \cdot \frac{d}{dx} \left( x^3 \right) = e^{x^3} \cdot 3x^2 = 3x^2 e^{x^3} \]

Question 4:
Differentiate the following w.r.t. \( x \):

\( \sin(\tan^{-1} e^{-x}) \)
Let $y = \sin(\tan^{-1} e^x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin(\tan^{-1} e^x)\right]$$

$$= \cos(\tan^{-1} e^x) \cdot \frac{d}{dx}(\tan^{-1} e^x)$$

$$= \cos(\tan^{-1} e^x) \cdot \frac{1}{1 + (e^x)^2} \cdot \frac{d}{dx}(e^x)$$

$$= \frac{\cos(\tan^{-1} e^x) \cdot e^x}{1 + e^{-2x}} \cdot e^x \cdot \frac{d}{dx}(-x)$$

$$= \frac{e^{-x} \cos(\tan^{-1} e^x)}{1 + e^{-2x}} \times (-1)$$

$$= \frac{-e^{-x} \cos(\tan^{-1} e^x)}{1 + e^{-2x}}$$

**Question 5:**

Differentiate the following w.r.t. $x$:

$log(cos e^x)$

**Answer**

Let $y = log(cos e^x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[log(cos e^x)\right]$$

$$= \frac{1}{\cos e^x} \cdot \frac{d}{dx}(\cos e^x)$$

$$= \frac{1}{\cos e^x} \cdot (-\sin e^x) \cdot \frac{d}{dx}(e^x)$$

$$= -\sin e^x \cdot e^x$$

$$= -e^x \tan e^x \cdot e^x \cdot (2n+1) \frac{\pi}{2}, n \in \mathbb{N}$$
Question 6:
Differentiate the following w.r.t. $x$:

$$e^x + e^{x^2} + \ldots + e^{x^n}$$

Answer

$$\frac{d}{dx}(e^x + e^{x^2} + \ldots + e^{x^n})$$

$$= \frac{d}{dx}(e^x) + \frac{d}{dx}(e^{x^2}) + \frac{d}{dx}(e^{x^3}) + \ldots + \frac{d}{dx}(e^{x^n})$$

$$= e^x + [e^{x^2} \cdot \frac{d}{dx}(x^2)] + [e^{x^3} \cdot \frac{d}{dx}(x^3)] + \ldots + [e^{x^n} \cdot \frac{d}{dx}(x^n)]$$

$$= e^x + (e^{x^2} \cdot 2x) + (e^{x^3} \cdot 3x^2) + \ldots + (e^{x^n} \cdot nx^{n-1})$$

$$= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + \ldots + 5x^ne^{x^n}$$

Question 7:
Differentiate the following w.r.t. $x$:

$$\sqrt{e^{\sqrt{x}}}, x > 0$$

Answer

Let $y = \sqrt{e^{\sqrt{x}}}$

Then, $y^2 = e^{\sqrt{x}}$

By differentiating this relationship with respect to $x$, we obtain
Question 8:
Differentiate the following w.r.t. $x$:

$$\log(\log x), x > 1$$

Answer

Let $y = \log(\log x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\log x)\right]
= \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)
= \frac{1}{\log x} \cdot \frac{1}{x}
= \frac{1}{x \log x}, x > 1$$

Question 9:
Differentiate the following w.r.t. $x$:

$$\frac{\cos x}{\log x}, x > 0$$
Answer

\[ y = \frac{\cos x}{\log x} \]

Let \[ y = \frac{\cos x}{\log x} \]

By using the quotient rule, we obtain

\[
\frac{dy}{dx} = \frac{\frac{d}{dx}(\cos x) \times \log x - \cos x \times \frac{d}{dx}(\log x)}{(\log x)^2}
\]

\[ = \frac{-\sin x \log x - \cos x \times \frac{1}{x}}{(\log x)^2} \]

\[ = \frac{-[x \log x \sin x + \cos x]}{x (\log x)^2}, \quad x > 0 \]

**Question 10:**

Differentiate the following w.r.t. \( x \):

\[ \cos \left( \log x + e^x \right), \quad x > 0 \]

**Answer**

Let \[ y = \cos \left( \log x + e^x \right) \]

By using the chain rule, we obtain

\[
\frac{dy}{dx} = -\sin \left( \log x + e^x \right) \cdot \frac{d}{dx} \left( \log x + e^x \right)
\]

\[ = -\sin \left( \log x + e^x \right) \cdot \left[ \frac{d}{dx} (\log x) + \frac{d}{dx} (e^x) \right] \]

\[ = -\sin \left( \log x + e^x \right) \cdot \left( \frac{1}{x} + e^x \right) \]

\[ = -\left( \frac{1}{x} + e^x \right) \sin \left( \log x + e^x \right), \quad x > 0 \]
Exercise 5.5

Question 1:
Differentiate the function with respect to $x$.

$$\cos x \cdot \cos 2x \cdot \cos 3x$$

Answer

Let $y = \cos x \cdot \cos 2x \cdot \cos 3x$

Taking logarithm on both the sides, we obtain

$$\log y = \log (\cos x \cdot \cos 2x \cdot \cos 3x)$$

$$\Rightarrow \log y = \log (\cos x) + \log (\cos 2x) + \log (\cos 3x)$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{\cos x} \frac{d}{dx} (\cos x) + \frac{1}{\cos 2x} \frac{d}{dx} (\cos 2x) + \frac{1}{\cos 3x} \frac{d}{dx} (\cos 3x)$$

$$\Rightarrow \frac{dy}{dx} = y \left[ -\frac{\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x} \cdot \frac{d}{dx} (2x) - \frac{\sin 3x}{\cos 3x} \cdot \frac{d}{dx} (3x) \right]$$

$$\therefore \frac{dy}{dx} = -\cos x \cdot \cos 2x \cdot \cos 3x \left[ \tan x + 2 \tan 2x + 3 \tan 3x \right]$$

Question 2:
Differentiate the function with respect to $x$.

$$\frac{(x-1)(x-2)}{\sqrt{(x-3)(x-4)(x-5)}}$$

Answer

Let $y = \frac{(x-1)(x-2)}{\sqrt{(x-3)(x-4)(x-5)}}$

Taking logarithm on both the sides, we obtain

$$\log y = \log \left( \frac{(x-1)(x-2)}{\sqrt{(x-3)(x-4)(x-5)}} \right)$$

$$\Rightarrow \log y = \log (x-1) + \log (x-2) - \frac{1}{2} \left( \log (x-3) + \log (x-4) + \log (x-5) \right)$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{2} \left( \frac{1}{x-3} + \frac{1}{x-4} + \frac{1}{x-5} \right) \cdot \frac{d}{dx} \left( (x-3)(x-4)(x-5) \right)$$

$$\Rightarrow \frac{dy}{dx} = y \left[ \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{2} \left( \frac{1}{x-3} + \frac{1}{x-4} + \frac{1}{x-5} \right) \right] \cdot \frac{d}{dx} \left( (x-3)(x-4)(x-5) \right)$$

$$\therefore \frac{dy}{dx} = \frac{(x-1)(x-2)}{\sqrt{(x-3)(x-4)(x-5)}} \left[ \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{2} \left( \frac{1}{x-3} + \frac{1}{x-4} + \frac{1}{x-5} \right) \right] \cdot \frac{d}{dx} \left( (x-3)(x-4)(x-5) \right)$$
Differentiating both sides with respect to \( x \), we obtain

\[
\frac{1}{y} \frac{dy}{dx} = \frac{1}{x-1} \frac{d}{dx} (x-1) + \frac{1}{x-2} \frac{d}{dx} (x-2) - \frac{1}{x-3} \frac{d}{dx} (x-3) - \frac{1}{x-4} \frac{d}{dx} (x-4) - \frac{1}{x-5} \frac{d}{dx} (x-5)
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{y}{2} \left[ \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right]
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{y}{2} \left[ \frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)} \left[ \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right] \right]
\]

**Question 3:**
Differentiate the function with respect to \( x \).

\[(\log x)^{\cos x}\]

**Answer**
Let \( y = (\log x)^{\cos x} \)

Taking logarithm on both the sides, we obtain

\[
\log y = \cos x \cdot \log (\log x)
\]

Differentiating both sides with respect to \( x \), we obtain
Question 4:
Differentiate the function with respect to \( x \).

\[ x^x - 2^\sin x \]

Answer

Let \( y = x^x - 2^\sin x \)

Also, let \( x^x = u \) and \( 2^\sin x = v \)

\[ \therefore y = u - v \]

\[ \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \]

\[ u = x^x \]

Taking logarithm on both the sides, we obtain

\[ \log u = x \log x \]

Differentiating both sides with respect to \( x \), we obtain

\[ \frac{1}{u} \frac{du}{dx} = \left[ \frac{d}{dx} (x) \times \log x + x \times \frac{d}{dx} (\log x) \right] \]

\[ \Rightarrow \frac{du}{dx} = u \left[ 1 \times \log x + x \times \frac{1}{x} \right] \]

\[ \Rightarrow \frac{du}{dx} = x^x (\log x + 1) \]

\[ \Rightarrow \frac{dv}{dx} = x^x (1 + \log x) \]

\[ v = 2^\sin x \]

Taking logarithm on both the sides with respect to \( x \), we obtain
\[
\log v = \sin x \cdot \log 2
\]

Differentiating both sides with respect to \(x\), we obtain

\[
\frac{1}{v} \cdot \frac{dv}{dx} = \log 2 \cdot \frac{d}{dx} (\sin x)
\]

\[
\Rightarrow \frac{dv}{dx} = v \log 2 \cos x
\]

\[
\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2
\]

\[
\therefore \frac{dy}{dx} = x^x (1 + \log x) - 2^{\sin x} \cos x \log 2
\]

**Question 5:**

Differentiate the function with respect to \(x\).

\[
(x + 3)^2 \cdot (x + 4)^3 \cdot (x + 5)^4
\]

**Answer**

Let \(y = (x + 3)^2 \cdot (x + 4)^3 \cdot (x + 5)^4\)

Taking logarithm on both the sides, we obtain

\[
\log y = \log (x + 3)^2 + \log (x + 4)^3 + \log (x + 5)^4
\]

\[
\Rightarrow \log y = 2 \log (x + 3) + 3 \log (x + 4) + 4 \log (x + 5)
\]

Differentiating both sides with respect to \(x\), we obtain

\[
\frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \frac{1}{x + 3} \cdot \frac{d}{dx} (x + 3) + 3 \cdot \frac{1}{x + 4} \cdot \frac{d}{dx} (x + 4) + 4 \cdot \frac{1}{x + 5} \cdot \frac{d}{dx} (x + 5)
\]

\[
\Rightarrow \frac{dy}{dx} = y \left[ \frac{2}{x + 3} + \frac{3}{x + 4} + \frac{4}{x + 5} \right]
\]

\[
\Rightarrow \frac{dy}{dx} = (x + 3)^2 \cdot (x + 4)^3 \cdot (x + 5)^4 \cdot \left[ \frac{2}{x + 3} + \frac{3}{x + 4} + \frac{4}{x + 5} \right]
\]

\[
\Rightarrow \frac{dy}{dx} = (x + 3)^2 \cdot (x + 4)^3 \cdot (x + 5)^4 \cdot \left[ \frac{2(x + 4)(x + 5) + 3(x + 3)(x + 5) + 4(x + 3)(x + 4)}{(x + 3)(x + 4)(x + 5)} \right]
\]

\[
\Rightarrow \frac{dy}{dx} = (x + 3)(x + 4)^2 \cdot (x + 5)^3 \cdot \left[ 2(x^2 + 9x + 20) + 3(x^2 + 8x + 15) + 4(x^2 + 7x + 12) \right]
\]

\[
\therefore \frac{dy}{dx} = (x + 3)(x + 4)^2 \cdot (x + 5)^3 \cdot (9x^2 + 70x + 133)
\]
Question 6:
Differentiate the function with respect to \( x \).

\[
\left( x + \frac{1}{x} \right)^x + x^{\left( \frac{x+1}{x} \right)}
\]

Answer

Let \( y = \left( x + \frac{1}{x} \right)^x + x^{\left( \frac{x+1}{x} \right)} \)

Also, let \( u = \left( x + \frac{1}{x} \right)^x \) and \( v = x^{\left( \frac{x+1}{x} \right)} \)

\[ \therefore y = u + v \]

\[ \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(1)} \]

Then, \( u = \left( x + \frac{1}{x} \right)^x \)

\[ \Rightarrow \log u = \log \left( x + \frac{1}{x} \right)^x \]

\[ \Rightarrow \log u = x \log \left( x + \frac{1}{x} \right) \]

Differentiating both sides with respect to \( x \), we obtain
\[ \frac{1}{u} \frac{du}{dx} = \frac{d}{dx} \log \left( x + \frac{1}{x} \right) + x \frac{d}{dx} \left[ \log \left( x + \frac{1}{x} \right) \right] \]

\[ \Rightarrow \frac{1}{u} \frac{du}{dx} = 1 \frac{d}{dx} \log \left( x + \frac{1}{x} \right) + x \frac{1}{x + \frac{1}{x}} \frac{d}{dx} \left( x + \frac{1}{x} \right) \]

\[ \Rightarrow \frac{du}{dx} = u \left[ \log \left( x + \frac{1}{x} \right) + \frac{x}{x + \frac{1}{x}} \times \left( x - \frac{1}{x} \right) \right] \]

\[ \Rightarrow \frac{du}{dx} = \left( x + \frac{1}{x} \right)^x \left[ \log \left( x + \frac{1}{x} \right) + \frac{x - \frac{1}{x}}{x + \frac{1}{x}} \right] \]

\[ \Rightarrow \frac{du}{dx} = \left( x + \frac{1}{x} \right)^x \left[ \log \left( x + \frac{1}{x} \right) + \frac{x^2 - 1}{x^2 + 1} \right] \]

\[ \Rightarrow \frac{du}{dx} = \left( x + \frac{1}{x} \right)^x \left[ \frac{x^2 - 1}{x^2 + 1} + \log \left( x + \frac{1}{x} \right) \right] \quad ...(2) \]

\[ v = x^{\frac{1}{x}} \]

\[ \Rightarrow \log v = \log \left[ x^{\frac{1}{x}} \right] \]

\[ \Rightarrow \log v = \left( 1 + \frac{1}{x} \right) \log x \]

Differentiating both sides with respect to \( x \), we obtain
\[
\frac{1}{v} \frac{dv}{dx} = \left[ \frac{d}{dx} \left( \frac{1 + \frac{1}{x}}{x} \right) \right] \times \log x + \left( 1 + \frac{1}{x} \right) \frac{d}{dx} \log x
\]

\[\Rightarrow \frac{1}{v} \frac{dv}{dx} = \left( - \frac{1}{x^2} \right) \log x + \left( 1 + \frac{1}{x} \right) \frac{1}{x}
\]

\[\Rightarrow \frac{1}{v} \frac{dv}{dx} = - \frac{\log x + 1 + \frac{1}{x}}{x^2} \]

\[\Rightarrow \frac{dv}{dx} = v \left[ - \frac{\log x + x + 1}{x^2} \right]
\]

\[\Rightarrow \frac{dv}{dx} = x \left( ^{1+1} \middle/ x \right) \left( \frac{x + 1 - \log x}{x^2} \right) \quad \ldots (3)
\]

Therefore, from (1), (2), and (3), we obtain

\[\frac{dv}{dx} = \left( x + \frac{1}{x} \right) \left[ \frac{x^2 - 1}{x^2 + 1} + \log \left( x + \frac{1}{x} \right) \right] + x \left[ ^{1+1} \middle/ x \right] \left( \frac{x + 1 - \log x}{x^2} \right) \]

**Question 7:**

Differentiate the function with respect to \(x\).

\[(\log x)^x + x^{\log x}\]

**Answer**

Let \(y = (\log x)^x + x^{\log x}\)

Also, let \(u = (\log x)^x\) and \(v = x^{\log x}\)

\[\Rightarrow y = u + v\]

\[\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots (1)\]

\(u = (\log x)^x\)

\[\Rightarrow \log u = \log \left( (\log x)^x \right)\]

\[\Rightarrow \log u = x \log (\log x)\]

Differentiating both sides with respect to \(x\), we obtain
\[
\frac{1}{u} \frac{du}{dx} = \frac{d}{dx} \left( x \cdot \log(\log x) + x \cdot \frac{d}{dx}[\log(\log x)] \right)
\]
\[
\Rightarrow \frac{du}{dx} = u \left[ 1 \cdot \log(\log x) + x \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) \right]
\]
\[
\Rightarrow \frac{du}{dx} = (\log x)^x \left[ \log(\log x) + \frac{x}{\log x} \cdot \frac{1}{x} \right]
\]
\[
\Rightarrow \frac{du}{dx} = (\log x)^x \left[ \log(\log x) + \frac{1}{\log x} \right]
\]
\[
\Rightarrow \frac{du}{dx} = (\log x)^x \left[ \log(\log x) \cdot \frac{\log x + 1}{\log x} \right]
\]
\[
\Rightarrow \frac{du}{dx} = (\log x)^{x-1} \left[ 1 + \log x \cdot \log(\log x) \right] \quad \text{...(2)}
\]

\[v = x^{\log x}\]
\[
\Rightarrow \log v = \log \left( x^{\log x} \right)
\]
\[
\Rightarrow \log v = \log x \cdot \log x = (\log x)^2
\]

Differentiating both sides with respect to \(x\), we obtain
\[
\frac{1}{v} \cdot \frac{dv}{dx} = \frac{d}{dx} \left[ (\log x)^2 \right]
\]
\[
\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = 2(\log x) \cdot \frac{d}{dx}(\log x)
\]
\[
\Rightarrow \frac{dv}{dx} = 2v(\log x) \cdot \frac{1}{x}
\]
\[
\Rightarrow \frac{dv}{dx} = 2x^{\log x} \cdot \frac{\log x}{x}
\]
\[
\Rightarrow \frac{dv}{dx} = 2x^{\log x - 1} \cdot \log x \quad \text{...(3)}
\]

Therefore, from (1), (2), and (3), we obtain
\[
\frac{dy}{dx} = (\log x)^{x-1} \left[ 1 + \log x \cdot \log(\log x) \right] + 2x^{\log x - 1} \cdot \log x
\]

Question 8:
Differentiate the function with respect to \(x\).
\[(\sin x)^x + \sin^{-1} \sqrt{x}\]

**Answer**

Let \(y = (\sin x)^x + \sin^{-1} \sqrt{x}\)

Also, let \(u = (\sin x)^x\) and \(v = \sin^{-1} \sqrt{x}\)

\[
\therefore \quad y = u + v
\]

\[
\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \cdots (1)
\]

\[
u = (\sin x)^x
\]

\[
\Rightarrow \quad \log u = \log(\sin x)^x
\]

\[
\Rightarrow \quad \log u = x \log(\sin x)
\]

Differentiating both sides with respect to \(x\), we obtain

\[
\Rightarrow \quad \frac{1}{u} \frac{du}{dx} = \frac{x}{\sin x} \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x)
\]

\[
\Rightarrow \quad \frac{du}{dx} = u \left[ 1 \cdot \log(\sin x) + \frac{x}{\sin x} \cdot \frac{d}{dx}(\sin x) \right]
\]

\[
\Rightarrow \quad \frac{du}{dx} = (\sin x)^x \left[ \log(\sin x) + \frac{x}{\sin x} \cdot \cos x \right]
\]

\[
\Rightarrow \quad \frac{du}{dx} = (\sin x)^x \left( x \cot x + \log \sin x \right) \quad \cdots (2)
\]

\[
v = \sin^{-1} \sqrt{x}
\]

Differentiating both sides with respect to \(x\), we obtain

\[
\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(\sqrt{x})
\]

\[
\Rightarrow \quad \frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}
\]

\[
\Rightarrow \quad \frac{dv}{dx} = \frac{1}{2\sqrt{x} - x^2} \quad \cdots (3)
\]

Therefore, from (1), (2), and (3), we obtain

\[
\frac{dy}{dx} = (\sin x)^x \left( x \cot x + \log \sin x \right) + \frac{1}{2\sqrt{x} - x^2}
\]
Question 9:
Differentiate the function with respect to $x$.

\[ x^{\sin x} + (\sin x)^{\cos x} \]

Answer

Let $y = x^{\sin x} + (\sin x)^{\cos x}$

Also, let $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$

\[ \therefore y = u + v \]

\[ \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \]

\[ u = x^{\sin x} \]

\[ \Rightarrow \log u = \log (x^{\sin x}) \]

\[ \Rightarrow \log u = \sin x \log x \]

Differentiating both sides with respect to $x$, we obtain

\[ \frac{1}{u} \frac{du}{dx} = \frac{d}{dx} (\sin x) \cdot \log x + \sin x \cdot \frac{d}{dx} (\log x) \]

\[ \Rightarrow \frac{du}{dx} = u \left[ \cos x \log x + \sin x \cdot \frac{1}{x} \right] \]

\[ \Rightarrow \frac{du}{dx} = x^{\sin x} \left[ \cos x \log x + \frac{\sin x}{x} \right] \] \hspace{1cm} ...(2)

\[ v = (\sin x)^{\cos x} \]

\[ \Rightarrow \log v = \log (\sin x)^{\cos x} \]

\[ \Rightarrow \log v = \cos x \log (\sin x) \]

Differentiating both sides with respect to $x$, we obtain
From (1), (2), and (3), we obtain

\[
\frac{dy}{dx} = \sin x \left( \cos x \log x + \frac{\sin x}{x} \right) + \left( \sin x \cos x - \sin x \log x \right)
\]

Question 10:
Differentiate the function with respect to \(x\).

\[x^{\cos x} + \frac{x^2 + 1}{x^2 - 1}\]

Answer

Let \(y = x^{\cos x} + \frac{x^2 + 1}{x^2 - 1}\)

Also, let \(u = x^{\cos x}\) and \(v = \frac{x^2 + 1}{x^2 - 1}\)

\[\therefore y = u + v\]

\[\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}\]

\[u = x^{\cos x}\]

\[\Rightarrow \log u = \log \left( x^{\cos x} \right)\]

\[\Rightarrow \log u = x \cos x \log x\]

Differentiating both sides with respect to \(x\), we obtain
\[
\frac{1}{u} \frac{du}{dx} = \frac{d}{dx}(x \cdot \cos x \cdot \log x + x \cdot \frac{d}{dx}(\cos x) \cdot \log x + x \cos x \cdot \frac{d}{dx}(\log x)
\]

\Rightarrow \frac{du}{dx} = u \left[ 1 \cdot \cos x \cdot \log x + x \cdot (-\sin x) \log x + x \cos x \cdot \frac{1}{x} \right]

\Rightarrow \frac{du}{dx} = x^{\cos x} \left( \cos x \log x - x \sin x \log x + x \cos x \right)

\Rightarrow \frac{du}{dx} = x^{\cos x} \left[ \cos x (1 + \log x) - x \sin x \log x \right] \quad \cdots (2)

v = \frac{x^2 + 1}{x^2 - 1}

\Rightarrow \log v = \log \left( \frac{x^2 + 1}{x^2 - 1} \right)

Differentiating both sides with respect to \( x \), we obtain

\[
\frac{1}{v} \frac{dv}{dx} = \frac{2x}{x^2 + 1} - \frac{2x}{x^2 - 1}
\]

\Rightarrow \frac{dv}{dx} = v \left[ \frac{2x(x^2 - 1) - 2x(x^2 + 1)}{(x^2 + 1)(x^2 - 1)} \right]

\Rightarrow \frac{dv}{dx} = \frac{x^2 + 1}{x^2 - 1} \times \left[ \frac{-4x}{(x^2 + 1)(x^2 - 1)} \right]

\Rightarrow \frac{dv}{dx} = \frac{-4x}{(x^2 - 1)^2} \quad \cdots (3)

From (1), (2), and (3), we obtain

\[
\frac{dy}{dx} = x^{\cos x} \left[ \cos x (1 + \log x) - x \sin x \log x \right] - \frac{4x}{(x^2 - 1)^2}
\]

Question 11:

Differentiate the function with respect to \( x \).

\[
(x \cos x)^y + (x \sin x)^{\frac{1}{x}}
\]
Answer

Let \( y = (\cos x)^x + x \sin x^x \)

Also, let \( u = (\cos x)^x \) and \( v = (x \sin x)^x \)

\[ \therefore y = u + v \]

\[ \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \]  \hspace{1cm} ...(1)

\[ u = (\cos x)^x \]

\[ \Rightarrow \log u = \log (\cos x)^x \]

\[ \Rightarrow \log u = x \log (\cos x) \]

\[ \Rightarrow \log u = x \left[ \log x + \log \cos x \right] \]

\[ \Rightarrow \log u = x \log x + x \log \cos x \]

Differentiating both sides with respect to \( x \), we obtain

\[ \frac{1}{u} \frac{du}{dx} = \frac{d}{dx} \left( x \log x \right) + \frac{d}{dx} \left( x \log \cos x \right) \]

\[ \Rightarrow \frac{du}{dx} = u \left[ \left( \log x \cdot \frac{d}{dx} (x) + x \cdot \frac{d}{dx} (\log x) \right) + \left( \log \cos x \cdot \frac{d}{dx} (x) + x \cdot \frac{d}{dx} (\log \cos x) \right) \right] \]

\[ \Rightarrow \frac{du}{dx} = (\cos x)^x \left[ \left( \log x + 1 + x \frac{1}{x} \right) + \left( \log \cos x - 1 + x \frac{1}{\cos x} \cdot \frac{d}{dx} (\cos x) \right) \right] \]

\[ \Rightarrow \frac{du}{dx} = (\cos x)^x \left[ \left( \log x + 1 \right) + \left( \log \cos x + x \frac{1}{\cos x} \cdot (\cos x - 1 + \tan x) \right) \right] \]

\[ \Rightarrow \frac{du}{dx} = (\cos x)^x \left[ \left( 1 + \log x \right) + \left( \log x + \log \cos x - x \tan x \right) \right] \]

\[ \Rightarrow \frac{du}{dx} = (\cos x)^x \left[ 1 - x \tan x + \log x + \log \cos x \right] \]

\[ \Rightarrow \frac{du}{dx} = (\cos x)^x \left[ 1 - x \tan x + \log (x \cos x) \right] \]  \hspace{1cm} ...(2)
Differentiating both sides with respect to \( x \), we obtain

\[
\frac{1}{v} \frac{dv}{dx} = \frac{d}{dx} \left( \frac{1}{x} \log x \right) + \frac{d}{dx} \left[ \frac{1}{x} \log (\sin x) \right]
\]

\[
\Rightarrow \frac{1}{v} \frac{dv}{dx} = \log x \cdot \frac{d}{dx} \left( \frac{1}{x} \right) + \frac{1}{x} \cdot \frac{d}{dx} (\log x) + \left[ \log (\sin x) \cdot \frac{d}{dx} \left( \frac{1}{x} \right) + \frac{1}{x} \cdot \frac{d}{dx} (\log (\sin x)) \right]
\]

\[
\Rightarrow \frac{1}{v} \frac{dv}{dx} = \log x \cdot \left( -\frac{1}{x^2} \right) + \frac{1}{x} \cdot \frac{1}{x} + \left[ \log (\sin x) \left( -\frac{1}{x^2} \right) + \frac{1}{x} \cdot \frac{1}{\sin x} \cdot \frac{d}{dx} (\sin x) \right]
\]

\[
\Rightarrow \frac{1}{v} \frac{dv}{dx} = \frac{1}{x^2} \left( 1 - \log x \right) + \left[ -\frac{\log (\sin x)}{x^2} + \frac{1}{x \sin x} \cdot \cos x \right]
\]

\[
\Rightarrow \frac{dv}{dx} = (x \sin x)^{\frac{1}{2}} \left[ -\frac{1 - \log x}{x^2} - \frac{\log (\sin x) + x \cot x}{x^2} \right]
\]

\[
\Rightarrow \frac{dv}{dx} = (x \sin x)^{\frac{1}{2}} \left[ \frac{1 - \log x - \log (\sin x) + x \cot x}{x^2} \right]
\]

\[
\Rightarrow \frac{dv}{dx} = (x \sin x)^{\frac{1}{2}} \left[ \frac{1 - \log (x \sin x) + x \cot x}{x^2} \right] \quad \ldots (3)
\]

From (1), (2), and (3), we obtain

\[
\frac{dy}{dx} = (x \cos x)^{\frac{1}{2}} \left[ 1 - x \tan x + \log (x \cos x) \right] + (x \sin x)^{\frac{1}{2}} \left[ \frac{x \cot x + 1 - \log (x \sin x)}{x^2} \right]
\]

Question 12:

\[
\frac{dy}{dx}
\]

Find \( dx \) of function.
The given function is \( x^x + y^x = 1 \)

Let \( x^x = u \) and \( y^x = v \)

Then, the function becomes \( u + v = 1 \)

\[
\therefore \frac{du}{dx} + \frac{dv}{dx} = 0 \quad \ldots (1)
\]

\( u = x^x \)

\( \Rightarrow \log u = \log (x^x) \)

\( \Rightarrow \log u = y \log x \)

Differentiating both sides with respect to \( x \), we obtain

\[
\frac{1}{u} \cdot \frac{du}{dx} = \log x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx} \left( \log x \right)
\]

\[\Rightarrow \frac{du}{dx} = u \left[ \log x \cdot \frac{dy}{dx} + y \cdot \frac{1}{x} \right] \]

\[\Rightarrow \frac{du}{dx} = x^x \left( \log x \cdot \frac{dy}{dx} + \frac{y}{x} \right) \quad \ldots (2)\]

\( v = y^x \)

\( \Rightarrow \log v = \log (y^x) \)

\( \Rightarrow \log v = x \log y \)

Differentiating both sides with respect to \( x \), we obtain

\[
\frac{1}{v} \cdot \frac{dv}{dx} = \log y \cdot \frac{d}{dx} (x) + x \cdot \frac{d}{dx} \left( \log y \right)
\]

\[\Rightarrow \frac{dv}{dx} = v \left( \log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} \right) \]

\[\Rightarrow \frac{dv}{dx} = y^x \left( \log y + \frac{x}{y} \frac{dy}{dx} \right) \quad \ldots (3)\]

From (1), (2), and (3), we obtain
Question 13:

\[ \frac{dy}{dx} \]

Find \( \frac{dy}{dx} \) of function.

\[ y^r = x^r \]

Answer

The given function is \( y^r = x^r \)

Taking logarithm on both the sides, we obtain

\[ x \log y = y \log x \]

Differentiating both sides with respect to \( x \), we obtain

\[ \log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y) = \log x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(\log x) \]

\[ \Rightarrow \log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} = \log x \cdot \frac{dy}{dx} + y \cdot \frac{1}{x} \]

\[ \Rightarrow \log y + \frac{x}{y} \cdot \frac{dy}{dx} = \log x \cdot \frac{dy}{dx} + \frac{y}{x} \]

\[ \Rightarrow \left( \frac{x}{y} - \log x \right) \cdot \frac{dy}{dx} = \frac{y}{x} - \log y \]

\[ \Rightarrow \left( \frac{x - y \log x}{y} \right) \cdot \frac{dy}{dx} = \frac{y - x \log y}{x} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{y - x \log y}{x(1 - y \log x)} \]

Question 14:

\[ \frac{dy}{dx} \]

Find \( \frac{dy}{dx} \) of function.

\[ \frac{dy}{dx} = \frac{y - x \log y}{x(1 - y \log x)} \]
\[(\cos x)^y = (\cos y)^x\]

**Answer**

The given function is \[(\cos x)^y = (\cos y)^x\]

Taking logarithm on both the sides, we obtain
\[y \log \cos x = x \log \cos y\]

Differentiating both sides, we obtain
\[
\frac{dy}{dx} \log \cos x + y \cdot \frac{d}{dx} \log \cos x = x \cdot \frac{d}{dx} \log \cos y + \log \cos y \cdot \frac{d}{dx} \log \cos x
\]

\[
\Rightarrow \frac{dy}{dx} \log \cos x + y \cdot \frac{1}{\cos x} \cdot \frac{d}{dx} \cos x = x \cdot \frac{1}{\cos y} \cdot \frac{d}{dx} \cos y
\]

\[
\Rightarrow \frac{dy}{dx} \log \cos x + y \cdot (-\sin x) = x \cdot \frac{1}{\cos y} \cdot (-\sin y) \cdot \frac{dy}{dx}
\]

\[
\Rightarrow \frac{dy}{dx} \log \cos x - y \tan x = x \cdot \frac{1}{\cos y} \cdot (-\sin y) \cdot \frac{dy}{dx}
\]

\[
\Rightarrow \left(\frac{\log \cos x + x \tan y}{\cos y}\right) \frac{dy}{dx} = y \tan x + \log \cos y
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}
\]

**Question 15:**

\[
\frac{dy}{dx}
\]

Find \(\frac{dx}{dy}\) of function.
\[xy = e^{x-y}\]

**Answer**

The given function is \(xy = e^{x-y}\)

Taking logarithm on both the sides, we obtain
\[
\log(xy) = \log(e^{x-y})
\]

\[
\Rightarrow \log x + \log y = (x - y) \log e
\]

\[
\Rightarrow \log x + \log y = (x - y) \times 1
\]

\[
\Rightarrow \log x + \log y = x - y
\]
Differentiating both sides with respect to $x$, we obtain

$$\frac{d}{dx} (\log x) + \frac{d}{dx} (\log y) = \frac{d}{dx} (x) \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x} \frac{dy}{dx} + \frac{1}{y} \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \left( 1 + \frac{1}{y} \right) \frac{dy}{dx} = 1 - \frac{1}{x}$$

$$\Rightarrow \left( \frac{y+1}{y} \right) \frac{dy}{dx} = \frac{x-1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$$

**Question 16:**

Find the derivative of the function given by $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f''(1)$.

**Answer**

The given relationship is $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both the sides, we obtain

$$\log f(x) = \log (1+x) + \log (1+x^2) + \log (1+x^4) + \log (1+x^8)$$

Differentiating both sides with respect to $x$, we obtain
Differentiate \((x^5 - 5x + 8)(x^3 + 7x + 9)\) in three ways mentioned below

(i) By using product rule.
(ii) By expanding the product to obtain a single polynomial.
(iii) By logarithmic differentiation.

Do they all give the same answer?

Answer

Let \(y = (x^5 - 5x + 8)(x^3 + 7x + 9)\)

(i)
Let \( x^2 - 5x + 8 = u \) and \( x^3 + 7x + 9 = v \)

\[ \therefore y = uv \]

\[ \Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx} \quad \text{(By using product rule)} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left( x^2 - 5x + 8 \right) \cdot \left( x^3 + 7x + 9 \right) + \left( x^3 - 5x + 8 \right) \cdot \frac{d}{dx} \left( x^3 + 7x + 9 \right) \]

\[ \Rightarrow \frac{dy}{dx} = (2x - 5)(3x^2 + 7) + (x^3 + 7x + 9)(3x^2 + 7) \]

\[ \Rightarrow \frac{dy}{dx} = 2x(3x^2 + 7x + 9) + (3x^2 + 7)(3x^2 + 7) - 5x(3x^2 + 7) + 8(3x^2 + 7) \]

\[ \Rightarrow \frac{dy}{dx} = (2x^2 + 14x + 18) - 5x^3 - 35x - 45 + (3x^2 + 7) - 15x^2 - 35x + 24x^2 + 56 \]

\[ \therefore \frac{dy}{dx} = 5x^2 - 4x^3 - 4x - 11 \]

(ii)

\[ y = (x^2 - 5x + 8)(x^3 + 7x + 9) \]

\[ = x^2(x^3 + 7x + 9) - 5x(x^3 + 7x + 9) + 8(x^3 + 7x + 9) \]

\[ = x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72 \]

\[ = x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72 \]

\[ \therefore \frac{dy}{dx} = \frac{d}{dx} \left( x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72 \right) \]

\[ = \frac{d}{dx} (x^5) - 5 \frac{d}{dx} (x^4) + 15 \frac{d}{dx} (x^3) - 26 \frac{d}{dx} (x^2) + 11 \frac{d}{dx} (x) + \frac{d}{dx} (72) \]

\[ = 5x^4 - 5 \times 4x^3 + 15 \times 3x^2 - 26 \times 2x + 11 \times 1 + 0 \]

\[ = 5x^4 - 20x^3 + 45x^2 - 52x + 11 \]

(iii)

Taking logarithm on both the sides, we obtain

\[ \log y = \log (x^2 - 5x + 8) + \log (x^3 + 7x + 9) \]

Differentiating both sides with respect to \( x \), we obtain
From the above three observations, it can be concluded that all the results of \( \frac{dy}{dx} \) are same.

**Question 18:**

If \( u, v \) and \( w \) are functions of \( x \), then show that

\[
\frac{d}{dx}(u,v,w) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}
\]

in two ways—first by repeated application of product rule, second by logarithmic differentiation.

**Answer**

Let \( y = u(v,w) \)

By applying product rule, we obtain
\[
\frac{dy}{dx} = \frac{du}{dx} (v \cdot w) + u \cdot \frac{d}{dx} (v \cdot w)
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{du}{dx} v \cdot w + u \cdot \left[ \frac{dv}{dx} w + v \cdot \frac{dw}{dx} \right] \quad \text{(Again applying product rule)}
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{du}{dx} v \cdot w + u \cdot \frac{dv}{dx} w + u \cdot v \cdot \frac{dw}{dx}
\]

By taking logarithm on both sides of the equation \( y = u \cdot v \cdot w \), we obtain

\[
\log y = \log u + \log v + \log w
\]

Differentiating both sides with respect to \( x \), we obtain

\[
\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx} (\log u) + \frac{d}{dx} (\log v) + \frac{d}{dx} (\log w)
\]

\[
\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx}
\]

\[
\Rightarrow \frac{dy}{dx} = y \left[ \frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx} \right]
\]

\[
\Rightarrow \frac{dy}{dx} = u \cdot v \cdot w \left[ \frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx} \right]
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}
\]

\[
\therefore \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}
\]
Exercise 5.6

Question 1:
If \( x \) and \( y \) are connected parametrically by the equation, without eliminating the
\[
\frac{dy}{dx}
\]
parameter, find \( \frac{dx}{dx} \).
\( x = 2at^2, \ y = at^4 \)
Answer
The given equations are \( x = 2at^2 \) and \( y = at^4 \)

Then,
\[
\frac{dx}{dt} = \frac{d}{dt}(2at^2) = 2a \cdot \frac{d}{dt}(t^2) = 2a \cdot 2t = 4at
\]
\[
\frac{dy}{dt} = \frac{d}{dt}(at^4) = a \cdot \frac{d}{dt}(t^4) = a \cdot 4t^3 = 4at^3
\]

\[
\therefore \frac{dy}{dx} = \left( \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right) = \frac{4at^3}{4at} = t^2
\]

Question 2:
If \( x \) and \( y \) are connected parametrically by the equation, without eliminating the
\[
\frac{dy}{dx}
\]
parameter, find \( \frac{dx}{dx} \).
\( x = a \cos \theta, \ y = b \cos \theta \)
Answer
The given equations are \( x = a \cos \theta \) and \( y = b \cos \theta \)

Then,
\[
\frac{dx}{d\theta} = \frac{d}{d\theta}(a \cos \theta) = a(-\sin \theta) = -asin \theta
\]
\[
\frac{dy}{d\theta} = \frac{d}{d\theta}(b \cos \theta) = b(-\sin \theta) = -b \sin \theta
\]

\[
\therefore \frac{dy}{dx} = \left( \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \right) = \frac{-b \sin \theta}{-asin \theta} = \frac{b}{a}
\]
Question 3:
If \( x \) and \( y \) are connected parametrically by the equation, without eliminating the parameter, find \( \frac{dy}{dx} \).

\( x = \sin t, \ y = \cos 2t \)

Answer

The given equations are \( x = \sin t \) and \( y = \cos 2t \)

Then, \( \frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t \)

\( \frac{dy}{dt} = \frac{d}{dt}(\cos 2t) = -\sin 2t \cdot \frac{d}{dt}(2t) = -2\sin 2t \)

\( \therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sin 2t}{\cos t} = \frac{-2 \cdot 2t \sin t \cos t}{\cos t} = -4\sin t \)

Question 4:
If \( x \) and \( y \) are connected parametrically by the equation, without eliminating the parameter, find \( \frac{dy}{dx} \).

\( x = 4t, \ y = \frac{4}{t} \)

Answer

The given equations are \( x = 4t \) and \( y = \frac{4}{t} \)
\[
\frac{dx}{dt} = \frac{d}{dt}(4t) = 4 \\
\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right) = 4 \cdot \frac{d}{dt}\left(\frac{1}{t}\right) = 4 \cdot \left(-\frac{1}{t^2}\right) = -\frac{4}{t^2} \\
\Rightarrow \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-\frac{4}{t^2}}{4} = -\frac{1}{t^2}
\]

**Question 5:**

If \(x\) and \(y\) are connected parametrically by the equation, without eliminating the parameter, find \(\frac{dy}{dx}\).

\(x = \cos \theta - \cos 2\theta, \ y = \sin \theta - \sin 2\theta\)

**Answer**

The given equations are \(x = \cos \theta - \cos 2\theta\) and \(y = \sin \theta - \sin 2\theta\)

Then, \(\frac{dx}{d\theta} = \frac{d}{d\theta}(\cos \theta - \cos 2\theta) = \frac{d}{d\theta} (\cos \theta) - \frac{d}{d\theta}(\cos 2\theta)\)

\[= -\sin \theta - (-2 \sin 2\theta) = 2 \sin 2\theta - \sin \theta\]

\(\frac{dy}{d\theta} = \frac{d}{d\theta} (\sin \theta - \sin 2\theta) = \frac{d}{d\theta} (\sin \theta) - \frac{d}{d\theta}(\sin 2\theta)\)

\[= \cos \theta - 2 \cos 2\theta\]

\[\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\cos \theta - 2 \cos 2\theta}{2 \sin 2\theta - \sin \theta}\]

**Question 6:**

If \(x\) and \(y\) are connected parametrically by the equation, without eliminating the parameter, find \(\frac{dy}{dx}\).

\(x = a(\theta - \sin \theta), \ y = a(1 + \cos \theta)\)

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Answer

The given equations are \( x = a(\theta - \sin \theta) \) and \( y = a(1 + \cos \theta) \)

Then, \( \frac{dx}{d\theta} = a\left[ \frac{d}{d\theta}(\theta) - \frac{d}{d\theta}(\sin \theta) \right] = a(1 - \cos \theta) \)

\[ \frac{dy}{d\theta} = a\left[ \frac{d}{d\theta}(1) + \frac{d}{d\theta}(\cos \theta) \right] = a\left[0 + (-\sin \theta)\right] = -a \sin \theta \]

\[ \therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a \sin \theta}{a(1 - \cos \theta)} = \frac{-2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} = \frac{-\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = -\cot \frac{\theta}{2} \]

Question 7:
If \( x \) and \( y \) are connected parametrically by the equation, without eliminating the parameter, find \( \frac{dy}{dx} \).

\[ x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, \quad y = \frac{\cos^3 t}{\sqrt{\cos 2t}} \]

Answer

The given equations are \( x = \frac{\sin^3 t}{\sqrt{\cos 2t}} \) and \( y = \frac{\cos^3 t}{\sqrt{\cos 2t}} \)
Then, \( \frac{dx}{dt} = \frac{d}{dt} \left[ \frac{\sin^3 t}{\sqrt{\cos 2t}} \right] \)

\[
= \frac{\sqrt{\cos 2t} \cdot \frac{d}{dt} (\sin^3 t) - \sin^3 t \cdot \frac{d}{dt} \sqrt{\cos 2t}}{\cos 2t}
\]

\[
= \frac{\sqrt{\cos 2t} \cdot 3 \sin^2 t \cdot \frac{d}{dt} (\sin t) - \sin^3 t \cdot \frac{1}{2 \sqrt{\cos 2t}} \cdot \frac{d}{dt} (\cos 2t)}{\cos 2t}
\]

\[
= \frac{3 \sqrt{\cos 2t} \cdot \sin^2 t \cdot \cos t - \frac{\sin^3 t}{2 \sqrt{\cos 2t}} \cdot (-2 \sin 2t)}{\cos 2t}
\]

\[
= \frac{3 \cos 2t \sin^2 t \cos t + \sin^3 t \sin 2t}{\cos 2t \sqrt{\cos 2t}}
\]

Similarly,

\[
\frac{dy}{dt} = \frac{d}{dt} \left[ \frac{\cos^3 t}{\sqrt{\cos 2t}} \right]
\]

\[
= \frac{\sqrt{\cos 2t} \cdot \frac{d}{dt} (\cos^3 t) - \cos^3 t \cdot \frac{d}{dt} \sqrt{\cos 2t}}{\cos 2t}
\]

\[
= \frac{\sqrt{\cos 2t} \cdot 3 \cos^2 t \cdot \frac{d}{dt} (\cos t) - \cos^3 t \cdot \frac{1}{2 \sqrt{\cos 2t}} \cdot \frac{d}{dt} (\cos 2t)}{\cos 2t}
\]

\[
= \frac{3 \sqrt{\cos 2t} \cdot \cos^2 t \cdot (-\sin t) - \cos^3 t \cdot \frac{1}{2 \sqrt{\cos 2t}} \cdot (-2 \sin 2t)}{\cos 2t}
\]

\[
= \frac{-3 \cos 2t \cdot \cos^2 t \cdot \sin t + \cos^3 t \cdot \sin 2t}{\cos 2t \cdot \sqrt{\cos 2t}}
\]
\[
\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}
\]

\[
= \frac{3 \cos 2t \cos^2 t \sin t + \cos^3 t \sin 2t}{3 \cos 2t \sin^2 t \cos t + \sin^3 t \sin 2t}
\]

\[
= \frac{-3 \cos 2t \cos^2 t \sin t + \cos^3 t (2 \sin t \cos t)}{3 \cos 2t \sin^2 t \cos t + \sin^3 t (2 \sin t \cos t)}
\]

\[
= \frac{\sin t \cos t [-3 \cos 2t \cos t + 2 \cos^3 t]}{\sin t \cos t [3 \cos 2t \sin t + 2 \sin^3 t]}
\]

\[
= \frac{-3 (2 \cos^2 t - 1) \cos t + 2 \cos^3 t}{3 (1 - 2 \sin^2 t) \sin t + 2 \sin^3 t}
\]

\[
= \frac{-3 \cos^3 t + 3 \cos t}{3 \sin t - 4 \sin^3 t}
\]

\[
= \frac{-\cos 3t}{\sin 3t}
\]

\[
= -\cot 3t
\]

**Question 8:**

If \(x\) and \(y\) are connected parametrically by the equation, without eliminating the parameter, find \(dx\).

\[
x = a \left( \cos t + \log \tan \frac{t}{2} \right), \quad y = a \sin t
\]

**Answer**

\[
x = a \left( \cos t + \log \tan \frac{t}{2} \right) \quad \text{and} \quad y = a \sin t
\]

The given equations are
Question 9:

If \( x \) and \( y \) are connected parametrically by the equation, without eliminating the parameter, find \( \frac{dx}{dy} \).

\[
\frac{dx}{dy} = a \cdot \frac{d}{dt} \left( \frac{\cos t}{\sin t} \right) = a \cos t
\]

\[
\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a \cos t}{\cos^2 t} = \frac{\sin t}{\cos t} = \tan t
\]
Answer

The given equations are \( x = a \sec \theta \) and \( y = b \tan \theta \)

Then, \( \frac{dx}{d\theta} = a \cdot \frac{d}{d\theta} (\sec \theta) = a \sec \theta \tan \theta \)

\( \frac{dy}{d\theta} = b \cdot \frac{d}{d\theta} (\tan \theta) = b \sec^2 \theta \)

\( \therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b \sec^2 \theta}{a \sec \theta \tan \theta} = \frac{b \sec \theta \cot \theta}{a} = \frac{b \cos \theta}{a \cos \theta \sin \theta} = \frac{b}{a} \times \frac{1}{\sin \theta} = \frac{b}{a} \cosec \theta \)

Question 10:

If \( x \) and \( y \) are connected parametrically by the equation, without eliminating the parameter, find \( \frac{dy}{dx} \).

\( x = a(\cos \theta + \theta \sin \theta) \), \( y = a(\sin \theta - \theta \cos \theta) \)

Answer

The given equations are \( x = a(\cos \theta + \theta \sin \theta) \) and \( y = a(\sin \theta - \theta \cos \theta) \)

Then, \( \frac{dx}{d\theta} = a \left[ \frac{d}{d\theta} (\cos \theta) + \frac{d}{d\theta} (\theta \sin \theta) \right] = a \left[ -\sin \theta + \theta \frac{d}{d\theta} (\sin \theta) + \sin \theta \frac{d}{d\theta} (\theta) \right] \)

\( = a \left[ -\sin \theta + \theta \cos \theta + \sin \theta \right] = a \theta \cos \theta \)

\( \frac{dy}{d\theta} = a \left[ \frac{d}{d\theta} (\sin \theta) - \frac{d}{d\theta} (\theta \cos \theta) \right] = a \left[ \cos \theta - \left( \theta \frac{d}{d\theta} (\cos \theta) + \cos \theta \frac{d}{d\theta} (\theta) \right) \right] \)

\( = a \left[ \cos \theta + \theta \sin \theta - \cos \theta \right] \)

\( = a \theta \sin \theta \)

\( \therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a \theta \sin \theta}{a \theta \cos \theta} = \tan \theta \)

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Question 11:

If \( x = \sqrt{a^{\sin^{-1}t}} \), \( y = \sqrt{a^{\cos^{-1}t}} \), show that \( \frac{dy}{dx} = -\frac{y}{x} \)

Answer

The given equations are \( x = \sqrt{a^{\sin^{-1}t}} \) and \( y = \sqrt{a^{\cos^{-1}t}} \)

\( x = \sqrt{a^{\sin^{-1}t}} \) and \( y = \sqrt{a^{\cos^{-1}t}} \)

\( \Rightarrow x = \left(a^{\sin^{-1}t}\right)^{\frac{1}{2}} \) and \( y = \left(a^{\cos^{-1}t}\right)^{\frac{1}{2}} \)

\( \Rightarrow x = a^{\frac{1}{2}\sin^{-1}t} \) and \( y = a^{\frac{1}{2}\cos^{-1}t} \)

Consider \( x = a^{\frac{1}{2}\sin^{-1}t} \)

Taking logarithm on both the sides, we obtain

\( \log x = \frac{1}{2} \sin^{-1}t \log a \)

\( \therefore \frac{1}{x} \frac{dx}{dt} = \frac{1}{2} \log a \cdot \frac{d}{dt} \left(\sin^{-1}t\right) \)

\( \Rightarrow \frac{dx}{dt} = \frac{x}{2 \log a} \cdot \frac{1}{\sqrt{1-t^2}} \)

\( \Rightarrow \frac{dx}{dt} = \frac{x \log a}{2\sqrt{1-t^2}} \)

Then, consider \( y = a^{\frac{1}{2}\cos^{-1}t} \)

Taking logarithm on both the sides, we obtain

\( \log y = \frac{1}{2} \cos^{-1}t \log a \)

\( \therefore \frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \log a \cdot \frac{d}{dt} \left(\cos^{-1}t\right) \)

\( \Rightarrow \frac{dy}{dt} = y \log a \cdot \left(\frac{-1}{\sqrt{1-t^2}}\right) \)

\( \Rightarrow \frac{dy}{dt} = -\frac{y \log a}{\sqrt{1-t^2}} \)
\[ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-y \log a}{2\sqrt{1-t^2}} = -y \cdot x. \]

Hence, proved.
Exercise 5.7

Question 1:
Find the second order derivatives of the function.
\( x^2 + 3x + 2 \)
Answer
Let \( y = x^2 + 3x + 2 \)
Then,
\[
\frac{dy}{dx} = \frac{d}{dx} (x^2) + \frac{d}{dx} (3x) + \frac{d}{dx} (2) = 2x + 3 + 0 = 2x + 3
\]
\[
\therefore \frac{d^2y}{dx^2} = \frac{d}{dx} (2x + 3) = \frac{d}{dx} (2x) + \frac{d}{dx} (3) = 2 + 0 = 2
\]

Question 2:
Find the second order derivatives of the function.
\( x^{20} \)
Answer
Let \( y = x^{20} \)
Then,
\[
\frac{dy}{dx} = \frac{d}{dx} (x^{20}) = 20x^{19}
\]
\[
\therefore \frac{d^2y}{dx^2} = \frac{d}{dx} (20x^{19}) = 20 \frac{d}{dx} (x^{19}) = 20 \cdot 19 \cdot x^{18} = 380x^{18}
\]

Question 3:
Find the second order derivatives of the function.
\( x \cdot \cos x \)
Answer
Let \( y = x \cdot \cos x \)
Then,
\[ \frac{dy}{dx} = \frac{d}{dx} \left( x \cos x \right) = \cos x \frac{d}{dx} (x) + x \frac{d}{dx} (\cos x) = \cos x \cdot 1 + x \left( -\sin x \right) = \cos x - x \sin x \]

\[ \therefore \frac{d^2 y}{dx^2} = \frac{d}{dx} \left[ \cos x - x \sin x \right] = \frac{d}{dx} (\cos x) - \frac{d}{dx} (x \sin x) \]

\[ = -\sin x - \left[ \sin x \cdot \frac{d}{dx} (x) + x \cdot \frac{d}{dx} (\sin x) \right] \]

\[ = -\sin x - (\sin x + x \cos x) \]

\[ = -(\cos x + 2 \sin x) \]

**Question 4:**
Find the second order derivatives of the function.

\[ \log x \]

**Answer**

Let \( y = \log x \)

Then,

\[ \frac{dy}{dx} = \frac{d}{dx} \left( \log x \right) = \frac{1}{x} \]

\[ \therefore \frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{1}{x} \right) = -\frac{1}{x^2} \]

**Question 5:**
Find the second order derivatives of the function.

\[ x^3 \log x \]

**Answer**

Let \( y = x^3 \log x \)

Then,
Question 6:
Find the second order derivatives of the function.

\( e^x \sin 5x \)

**Answer**

Let \( y = e^x \sin 5x \)

\[
\frac{dy}{dx} = \frac{d}{dx} (e^x \sin 5x) = \sin 5x \cdot \frac{d}{dx} (e^x) + e^x \cdot \frac{d}{dx} (\sin 5x)
\]

\[
= \sin 5x \cdot e^x + e^x \cdot \cos 5x \cdot \frac{d}{dx} (5x) = e^x \sin 5x + e^x \cos 5x \cdot 5
\]

\[
= e^x (\sin 5x + 5 \cos 5x)
\]

\[
\therefore \frac{d^2y}{dx^2} = \frac{d}{dx} \left[ e^x (\sin 5x + 5 \cos 5x) \right]
\]

\[
= (\sin 5x + 5 \cos 5x) \cdot \frac{d}{dx} (e^x) + e^x \cdot \frac{d}{dx} (\sin 5x + 5 \cos 5x)
\]

\[
= (\sin 5x + 5 \cos 5x) e^x + e^x \left[ \cos 5x \cdot \frac{d}{dx} (5x) + 5 (- \sin 5x) \cdot \frac{d}{dx} (5x) \right]
\]

\[
= e^x (\sin 5x + 5 \cos 5x) + e^x (5 \cos 5x - 25 \sin 5x)
\]

Then,

\[
e^x (10 \cos 5x - 24 \sin 5x) = 2e^x (5 \cos 5x - 12 \sin 5x)
\]
Question 7:
Find the second order derivatives of the function.

\( e^{6x} \cos 3x \)

Answer

Let \( y = e^{6x} \cos 3x \)

Then,

\[
\frac{dy}{dx} = \frac{d}{dx} \left( e^{6x} \cdot \cos 3x \right) = \cos 3x \cdot \frac{d}{dx} \left( e^{6x} \right) + e^{6x} \cdot \frac{d}{dx} \left( \cos 3x \right)
\]

\[
= \cos 3x \cdot 6e^{6x} + e^{6x} \cdot (-\sin 3x) \cdot \frac{d}{dx} (3x)
\]

\[
= 6e^{6x} \cos 3x - 3e^{6x} \sin 3x \quad \quad \ldots (1)
\]

\[
\therefore \frac{d^2y}{dx^2} = \frac{d}{dx} \left( 6e^{6x} \cos 3x - 3e^{6x} \sin 3x \right) = 6 \cdot \frac{d}{dx} \left( e^{6x} \cos 3x \right) - 3 \cdot \frac{d}{dx} \left( e^{6x} \sin 3x \right)
\]

\[
= 6 \left[ 6e^{6x} \cos 3x - 3e^{6x} \sin 3x \right] - 3 \left[ \sin 3x \cdot \frac{d}{dx} (e^{6x}) + e^{6x} \cdot \frac{d}{dx} (\sin 3x) \right] \quad \quad \text{[Using (1)]}
\]

\[
= 36e^{6x} \cos 3x - 18e^{6x} \sin 3x - 3 \left[ \sin 3x \cdot e^{6x} \cdot 6 + e^{6x} \cdot 3 \cos 3x \cdot 3 \right]
\]

\[
= 36e^{6x} \cos 3x - 18e^{6x} \sin 3x - 9e^{6x} \cos 3x
\]

\[
= 27e^{6x} \cos 3x - 36e^{6x} \sin 3x
\]

\[
= 9e^{6x} \left( 3 \cos 3x - 4 \sin 3x \right)
\]

Question 8:
Find the second order derivatives of the function.

\( \tan^{-1} x \)

Answer

Let \( y = \tan^{-1} x \)

Then,
Question 9:
Find the second order derivatives of the function.

\( \log(\log x) \)

Answer

Let \( y = \log(\log x) \)

Then,

\[
\frac{dy}{dx} = \frac{d}{dx} \left[ \log(\log x) \right] = \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) = \frac{1}{x \log x} = (\log x)^{-1}
\]

\[
\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx} \left[ (\log x)^{-1} \right] = (-1) \cdot (\log x)^{-2} \cdot \frac{d}{dx}(\log x)
\]

\[
= \frac{-1}{(x \log x)^2} \left[ \log x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x) \right]
\]

\[
= \frac{-1}{(x \log x)^2} \left[ \log x \cdot 1 + x \cdot \frac{1}{x} \right] = \frac{-1 + \log x}{(x \log x)^2}
\]

Question 10:
Find the second order derivatives of the function.

\( \sin(\log x) \)

Answer

Let \( y = \sin(\log x) \)

Then,

\[
\frac{dy}{dx} = \frac{d}{dx} \left[ \sin(\log x) \right] = \cos(\log x) \cdot \frac{d}{dx}(\log x) = \cos(\log x) \cdot \frac{1}{x \log x}
\]

\[
\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx} \left[ \cos(\log x) \cdot \frac{1}{x \log x} \right] = \cos(\log x) \cdot \frac{d}{dx} \left[ \frac{1}{x \log x} \right]
\]

\[
= \cos(\log x) \cdot \left( \frac{d}{dx} \left[ -\frac{1}{x \log x} \right] \right) = \cos(\log x) \cdot \frac{d}{dx} \left[ -\frac{1}{x \log x} \right]
\]

\[
= \cos(\log x) \cdot \frac{d}{dx} \left[ -\frac{1}{x \log x} \right] = \cos(\log x) \cdot \frac{1}{x \log x} \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x)
\]

\[
= \cos(\log x) \cdot \frac{1}{x \log x} \left[ 1 - \log x \right] = \cos(\log x) \cdot \frac{1 - \log x}{x \log x}
\]
\[ \frac{dy}{dx} = \frac{d}{dx} \left[ \sin \left( \log x \right) \right] = \cos \left( \log x \right) \cdot \frac{d}{dx} \left( \log x \right) = \frac{\cos \left( \log x \right)}{x} \]

\[ \therefore \frac{d^2 y}{dx^2} = \frac{d}{dx} \left[ \frac{\cos \left( \log x \right)}{x} \right] \]

\[ = \frac{x \cdot \frac{d}{dx} \left[ \cos \left( \log x \right) \right] - \cos \left( \log x \right) \cdot \frac{d}{dx} (x)}{x^2} \]

\[ = \frac{x \cdot \left[ -\sin \left( \log x \right) \cdot \frac{d}{dx} \left( \log x \right) \right] - \cos \left( \log x \right) \cdot 1}{x^2} \]

\[ = \frac{-x \sin \left( \log x \right) \cdot \frac{1}{x} - \cos \left( \log x \right)}{x^2} \]

\[ = \frac{-x \sin \left( \log x \right) \cdot \frac{1}{x} - \cos \left( \log x \right)}{x^2} \]

\[ = \frac{- \left[ \sin \left( \log x \right) \cdot \frac{1}{x} + \cos \left( \log x \right) \right]}{x^2} \]

Question 11:

If \( y = 5 \cos x - 3 \sin x \), prove that \( \frac{d^2 y}{dx^2} + y = 0 \)

Answer

It is given that, \( y = 5 \cos x - 3 \sin x \)

Then,
\[
\frac{dy}{dx} = \frac{d}{dx} (5 \cos x) - \frac{d}{dx} (3 \sin x) = 5 \frac{d}{dx} (\cos x) - 3 \frac{d}{dx} (\sin x) \\
= 5(-\sin x) - 3 \cos x = -(5 \sin x + 3 \cos x)
\]

\[
\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx} \left[ - (5 \sin x + 3 \cos x) \right] \\
= - \left[ 5 \frac{d}{dx} (\sin x) + 3 \frac{d}{dx} (\cos x) \right] \\
= - \left[ 5 \cos x + 3(-\sin x) \right] \\
= - \left[ 5 \cos x - 3 \sin x \right] \\
= -y
\]

\[
\therefore \frac{d^2 y}{dx^2} + y = 0
\]

Hence, proved.

**Question 12:**

If \( y = \cos^{-1} x \), find \( \frac{d^3 y}{dx^3} \) in terms of \( y \) alone.

**Answer**

It is given that, \( y = \cos^{-1} x \)

Then,
\[
\frac{dy}{dx} = \frac{d}{dx} \left( \cos^{-1} x \right) = -\frac{1}{\sqrt{1-x^2}} = -(1-x^2)^{-\frac{1}{2}} \\
\frac{d^2 y}{dx^2} = \frac{d}{dx} \left[ -(1-x^2)^{-\frac{1}{2}} \right] \\
= -\left( -\frac{1}{2} \right) (1-x^2)^{-\frac{3}{2}} \cdot \frac{d}{dx} (1-x^2) \\
= \frac{1}{2\sqrt{(1-x^2)^3}} \cdot (-2x) \\
\Rightarrow \frac{d^2 y}{dx^2} = \frac{-x}{\sqrt{(1-x^2)^3}} \quad \ldots (i)
\]

\( y = \cos^{-1} x \Rightarrow x = \cos y \)

Putting \( x = \cos y \) in equation (i), we obtain

\[
\frac{d^2 y}{dx^2} = \frac{-\cos y}{\sqrt{(1-\cos^2 y)^3}} \\
\Rightarrow \frac{d^2 y}{dx^2} = \frac{-\cos y}{\sqrt{(\sin^2 y)^3}} \\
= \frac{-\cos y}{\sin^3 y} \\
= \frac{-\cos y \times 1}{\sin y \times \sin^2 y} \\
\Rightarrow \frac{d^2 y}{dx^2} = -\cot y \cdot \cosec^3 y
\]

**Question 13:**

If \( y = 3\cos(\log x) + 4\sin(\log x) \), show that \( x^2 y_2 + xy_1 + y = 0 \)

**Answer**

It is given that,

\( y = 3\cos(\log x) + 4\sin(\log x) \)

Then,
\[ \frac{dy}{dx} = 3 \left( \frac{d}{dx} \cos (\log x) \right) + 4 \left( \frac{d}{dx} \sin (\log x) \right) \]

\[ = 3 \left( -\sin (\log x) \cdot \frac{d}{dx} (\log x) \right) + 4 \left( \cos (\log x) \cdot \frac{d}{dx} (\log x) \right) \]

\[ \therefore \frac{dy}{dx} = \frac{-3\sin (\log x) \cdot \frac{d}{dx} (\log x) + 4\cos (\log x) \cdot \frac{d}{dx} (\log x)}{x} \]

\[ \therefore \frac{d^2 y}{dx^2} = \frac{x \left( 4\cos (\log x) - 3\sin (\log x) \right) - \left( 4\cos (\log x) - 3\sin (\log x) \right) \frac{d}{dx} (\log x)}{x^2} \]

\[ = \frac{x \left( 4\cos (\log x) - 3\sin (\log x) \right) - 4\cos (\log x) + 3\sin (\log x)}{x^2} \]

\[ = \frac{-4\sin (\log x) - 3\cos (\log x) - 4\cos (\log x) + 3\sin (\log x)}{x^2} \]

\[ = \frac{-4\sin (\log x) - 7\cos (\log x)}{x^2} \]

\[ \therefore x^2 \frac{d^2 y}{dx^2} + xy' + y = \frac{-4\sin (\log x) - 7\cos (\log x)}{x^2} \]

\[ = x^2 \left( \frac{-\sin (\log x) - 7\cos (\log x)}{x^2} \right) + x \left( \frac{4\cos (\log x) - 3\sin (\log x)}{x} \right) + 3\cos (\log x) + 4\sin (\log x) \]

\[ = -\sin (\log x) - 7\cos (\log x) + 4\cos (\log x) - 3\sin (\log x) + 3\cos (\log x) + 4\sin (\log x) \]

\[ = 0 \]

Hence, proved.

**Question 14:**

If \( y = Ae^{\ln x} + Be^{\ln x} \), show that \( \frac{d^2 y}{dx^2} - (m+n) \frac{dy}{dx} + mny = 0 \)
Answer

It is given that, \( y = Ae^{mx} + Be^{nx} \)

Then,
\[
\frac{dy}{dx} = A \cdot \frac{d}{dx}(e^{mx}) + B \cdot \frac{d}{dx}(e^{nx}) = A \cdot e^{mx} \cdot \frac{d}{dx}(mx) + B \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Am^2 e^{mx} + Bn^2 e^{nx}
\]
\[
\frac{d^2 y}{dx^2} = \frac{d}{dx}(Am^2 e^{mx} + Bn^2 e^{nx}) = Am \cdot \frac{d}{dx}(e^{mx}) + Bn \cdot \frac{d}{dx}(e^{nx})
\]
\[
= Am \cdot e^{mx} \cdot \frac{d}{dx}(mx) + Bn \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Am^2 e^{mx} + Bn^2 e^{nx}
\]
\[
\therefore \frac{d^2 y}{dx^2} - (m + n) \frac{dy}{dx} + mn y = Am^2 e^{mx} + Bn^2 e^{nx} - (m + n) \left( Am e^{mx} + Bn e^{nx} \right) + mn \left( Ae^{mx} + Be^{nx} \right)
\]
\[
= Am^2 e^{mx} + Bn^2 e^{nx} - Am e^{mx} - Bn e^{nx} - Am e^{mx} - Bn e^{nx} + Am e^{mx} + Bn e^{nx} = 0
\]
Hence, proved.

Question 15:

If \( y = 500e^{7x} + 600e^{-7x} \), show that \( \frac{d^2 y}{dx^2} = 49y \)

Answer

It is given that, \( y = 500e^{7x} + 600e^{-7x} \)

Then,
\[
\frac{dy}{dx} = 500 \cdot \frac{d}{dx}(e^{7x}) + 600 \cdot \frac{d}{dx}(e^{-7x})
\]
\[
= 500 \cdot e^{7x} \cdot \frac{d}{dx}(7x) + 600 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)
\]
\[
= 3500e^{7x} - 4200e^{-7x}
\]
\[
\therefore \frac{d^2y}{dx^2} = 3500 \cdot \frac{d}{dx}(e^{7x}) - 4200 \cdot \frac{d}{dx}(e^{-7x})
\]
\[
= 3500 \cdot e^{7x} \cdot \frac{d}{dx}(7x) - 4200 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)
\]
\[
= 7 \times 3500 \cdot e^{7x} + 7 \times 4200 \cdot e^{-7x}
\]
\[
= 49 \times 500e^{7x} + 49 \times 600e^{-7x}
\]
\[
= 49 \left( 500e^{7x} + 600e^{-7x} \right)
\]
\[
= 49y
\]

Hence, proved.

**Question 16:**

If \(e^y(x+1) = 1\), show that \(\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2\)

**Answer**

The given relationship is \(e^y(x+1) = 1\)

\[e^y(x+1) = 1\]

\[\Rightarrow e^y = \frac{1}{x+1}\]

Taking logarithm on both the sides, we obtain

\[y = \log \left(\frac{1}{x+1}\right)\]

Differentiating this relationship with respect to \(x\), we obtain...
\[
\frac{dy}{dx} = (x+1) \frac{d}{dx} \left( \frac{1}{x+1} \right) = (x+1) \cdot \frac{-1}{(x+1)^2} = \frac{-1}{x+1}
\]

\[
\therefore \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{-1}{x+1} \right) = \left( \frac{-1}{(x+1)^2} \right) = \frac{1}{(x+1)^2}
\]

\[
\Rightarrow \frac{d^2y}{dx^2} = \left( \frac{-1}{x+1} \right)^2
\]

\[
\Rightarrow \frac{d^2y}{dx^2} = \left( \frac{dy}{dx} \right)^2
\]

Hence, proved.

**Question 17:**

If \( y = \left( \tan^{-1} x \right)^2 \), show that \( (x^2+1) \cdot y_2 + 2x \cdot (x^2+1) \cdot y_1 = 2 \)

**Answer**

The given relationship is \( y = \left( \tan^{-1} x \right)^2 \)

Then,

\[
y_1 = 2 \tan^{-1} x \cdot \frac{d}{dx} \left( \tan^{-1} x \right)
\]

\[
\Rightarrow y_1 = 2 \tan^{-1} x \cdot \frac{1}{1+x^2}
\]

\[
\Rightarrow \left(1+x^2\right)y_1 = 2 \tan^{-1} x
\]

Again differentiating with respect to \( x \) on both the sides, we obtain

\[
(1+x^2) \cdot y_2 + 2x \cdot (1+x^2) \cdot y_1 = 2 \left( \frac{1}{1+x^2} \right)
\]

\[
\Rightarrow \left(1+x^2\right)^2 \cdot y_2 + 2x \left(1+x^2\right) \cdot y_1 = 2
\]

Hence, proved.
Exercise 5.8

Question 1:

Verify Rolle’s Theorem for the function \( f(x) = x^2 + 2x - 8 \), \( x \in [-4, 2] \)

Answer

The given function, \( f(x) = x^2 + 2x - 8 \), being a polynomial function, is continuous in \([-4, 2]\) and is differentiable in \((-4, 2)\).

\( f(-4) = (-4)^2 + 2(-4) - 8 = 16 - 8 - 8 = 0 \)

\( f(2) = (2)^2 + 2(2) - 8 = 4 + 4 - 8 = 0 \)

\[ \therefore f(-4) = f(2) = 0 \]

\( \Rightarrow \) The value of \( f(x) \) at \(-4\) and \(2\) coincides.

Rolle’s Theorem states that there is a point \( c \in (-4, 2) \) such that \( f'(c) = 0 \)

\( f(x) = x^2 + 2x - 8 \)

\( \Rightarrow f'(x) = 2x + 2 \)

\( \therefore f'(c) = 0 \)

\( \Rightarrow 2c + 2 = 0 \)

\( \Rightarrow c = -1, \text{ where } c = -1 \in (-4, 2) \)

Hence, Rolle’s Theorem is verified for the given function.
Question 2:
Examine if Rolle’s Theorem is applicable to any of the following functions. Can you say something about the converse of Rolle’s Theorem from these examples?

(i) \( f(x) = \lfloor x \rfloor \) for \( x \in [5, 9] \)
(ii) \( f(x) = \lceil x \rceil \) for \( x \in [-2, 2] \)
(iii) \( f(x) = x^2 - 1 \) for \( x \in [1, 2] \)

Answer

By Rolle’s Theorem, for a function \( f : [a, b] \rightarrow \mathbb{R} \), if
(a) \( f \) is continuous on \([a, b]\)
(b) \( f \) is differentiable on \((a, b)\)
(c) \( f(a) = f(b) \)

then, there exists some \( c \in (a, b) \) such that \( f'(c) = 0 \)

Therefore, Rolle’s Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) \( f(x) = \lfloor x \rfloor \) for \( x \in [5, 9] \)

It is evident that the given function \( f(x) \) is not continuous at every integral point.
In particular, \( f(x) \) is not continuous at \( x = 5 \) and \( x = 9 \)

\( \Rightarrow f(x) \) is not continuous in \([5, 9]\).

Also, \( f'(5) = \lfloor 5 \rfloor = 5 \) and \( f'(9) = \lfloor 9 \rfloor = 9 \)
\( \therefore f'(5) \neq f'(9) \)
The differentiability of \( f \) in \((5, 9)\) is checked as follows.
Let \( n \) be an integer such that \( n \in (5, 9) \).

The left hand limit of \( f \) at \( x = n \) is,
\[
\lim_{h \to 0} \frac{f(n + h) - f(n)}{h} = \lim_{h \to 0} \frac{[n + h] - [n]}{h} = \lim_{h \to 0} \frac{n - n}{h} = \lim_{h \to 0} \frac{-1}{h} = \infty
\]

The right hand limit of \( f \) at \( x = n \) is,
\[
\lim_{h \to 0} \frac{f(n + h) - f(n)}{h} = \lim_{h \to 0} \frac{[n + h] - [n]}{h} = \lim_{h \to 0} \frac{n - n}{h} = \lim_{h \to 0} 0 = 0
\]

Since the left and right hand limits of \( f \) at \( x = n \) are not equal, \( f \) is not differentiable at \( x = n \).

\( \therefore f \) is not differentiable in \( (5, 9) \).

It is observed that \( f \) does not satisfy all the conditions of the hypothesis of Rolle’s Theorem.

Hence, Rolle’s Theorem is not applicable for \( f(x) = [x] \) for \( x \in [5, 9] \).

(iii) \( f(x) = [x] \) for \( x \in [-2, 2] \)

It is evident that the given function \( f(x) \) is not continuous at every integral point.
In particular, \( f(x) \) is not continuous at \( x = -2 \) and \( x = 2 \)

\( \Rightarrow f(x) \) is not continuous in \( [-2, 2] \).

Also, \( f(-2) = [-2] = -2 \) and \( f(2) = [2] = 2 \)

\( \therefore f(-2) \neq f(2) \)

The differentiability of \( f \) in \( (-2, 2) \) is checked as follows.
Let \( n \) be an integer such that \( n \in (-2, 2) \).

The left hand limit of \( f \) at \( x = n \) is,
\[
\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{[n+h] - [n]}{h} = \lim_{h \to 0} \frac{n+1-n}{h} = \lim_{h \to 0} \frac{-1}{h} = \infty
\]
The right hand limit of \( f \) at \( x = n \) is,
\[
\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{[n+h] - [n]}{h} = \lim_{h \to 0} \frac{n-n}{h} = \lim_{h \to 0} 0 = 0
\]
Since the left and right hand limits of \( f \) at \( x = n \) are not equal, \( f \) is not differentiable at \( x = n \)

\( \therefore \) \( f \) is not differentiable in \((-2, 2)\).

It is observed that \( f \) does not satisfy all the conditions of the hypothesis of Rolle’s Theorem.

Hence, Rolle’s Theorem is not applicable for \( f(x) = \lfloor x \rfloor \) for \( x \in [-2, 2] \).

(iii) \( f(x) = x^3 - 1 \) for \( x \in [1, 2] \)
It is evident that \( f \), being a polynomial function, is continuous in \([1, 2]\) and is differentiable in \((1, 2)\).
\[
f(1) = (1)^3 - 1 = 0
\]
\[
f(2) = (2)^3 - 1 = 3
\]
\( \therefore f(1) \neq f(2) \)

It is observed that \( f \) does not satisfy a condition of the hypothesis of Rolle’s Theorem.
Hence, Rolle’s Theorem is not applicable for \( f(x) = x^2 - 1 \) for \( x \in [1, 2] \).

**Question 3:**

If \( f: [-5, 5] \to \mathbb{R} \) is a differentiable function and if \( f'(x) \) does not vanish anywhere, then prove that \( f(-5) \neq f(5) \).

**Answer**

It is given that \( f: [-5, 5] \to \mathbb{R} \) is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) \( f \) is continuous on \([-5, 5]\).

(b) \( f \) is differentiable on \((-5, 5)\).

Therefore, by the Mean Value Theorem, there exists \( c \in (-5, 5) \) such that

\[
f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}
\]

\( \Rightarrow 10f'(c) = f(5) - f(-5) \)

It is also given that \( f'(x) \) does not vanish anywhere.

\( \therefore f'(c) \neq 0 \)

\( \Rightarrow 10f'(c) \neq 0 \)

\( \Rightarrow f(5) - f(-5) \neq 0 \)

\( \Rightarrow f(5) \neq f(-5) \)

Hence, proved.

**Question 4:**

Verify Mean Value Theorem, if \( f(x) = x^2 - 4x - 3 \) in the interval \([a, b]\), where \( a = 1 \) and \( b = 4 \).
Answer

The given function is \( f(x) = x^2 - 4x - 3 \)

\( f \), being a polynomial function, is continuous in \([1, 4]\) and is differentiable in \((1, 4)\) whose derivative is \( 2x - 4 \).

\[
\begin{align*}
    f(1) &= 1^2 - 4\times1 - 3 = -6, \\
    f(4) &= 4^2 - 4\times4 - 3 = -3 \\

    \therefore \quad \frac{f(b) - f(a)}{b - a} &= \frac{f(4) - f(1)}{4 - 1} = \frac{-3 - (-6)}{3} = \frac{3}{3} = 1
\end{align*}
\]

Mean Value Theorem states that there is a point \( c \in (1, 4) \) such that \( f'(c) = 1 \)

\[
f'(c) = 1 \\
\Rightarrow 2c - 4 = 1 \\
\Rightarrow c = \frac{5}{2}, \text{ where } c = \frac{5}{2} \in (1, 4)
\]

Hence, Mean Value Theorem is verified for the given function.

Question 5:

Verify Mean Value Theorem, if \( f(x) = x^3 - 5x^2 - 3x \) in the interval \([a, b]\), where \( a = 1 \) and \( b = 3 \). Find all \( c \in (1, 3) \) for which \( f'(c) = 0 \)

Answer

The given function \( f \) is \( f(x) = x^3 - 5x^2 - 3x \)

\( f \), being a polynomial function, is continuous in \([1, 3]\) and is differentiable in \((1, 3)\) whose derivative is \( 3x^2 - 10x - 3 \).

\[
\begin{align*}
    f(1) &= 1^3 - 5\times1^2 - 3\times1 = -7, \\
    f(3) &= 3^3 - 5\times3^2 - 3\times3 = -27 \\

    \therefore \quad \frac{f(b) - f(a)}{b - a} &= \frac{f(3) - f(1)}{3 - 1} = \frac{-27 - (-7)}{3 - 1} = \frac{-10}{2} = -5
\end{align*}
\]
Mean Value Theorem states that there exist a point \( c \in (1, 3) \) such that \( f'(c) = -10 \)

\[
f'(c) = -10 \\
\Rightarrow 3c^2 - 10c - 3 = 10 \\
\Rightarrow 3c^2 - 10c + 7 = 0 \\
\Rightarrow 3c^2 - 3c - 7c + 7 = 0 \\
\Rightarrow 3c(c - 1) - 7(c - 1) = 0 \\
\Rightarrow (c - 1)(3c - 7) = 0 \\
\Rightarrow c = 1, \frac{7}{3}, \text{ where } c = \frac{7}{3} \in (1, 3)
\]

Hence, Mean Value Theorem is verified for the given function and \( c = \frac{7}{3} \in (1, 3) \) is the only point for which \( f'(c) = 0 \)

**Question 6:**
Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

**Answer**

Mean Value Theorem states that for a function \( f: [a, b] \rightarrow \mathbb{R} \), if

(a) \( f \) is continuous on \([a, b]\)
(b) \( f \) is differentiable on \((a, b)\)

then, there exists some \( c \in (a, b) \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.
(i) \( f(x) = [x] \) for \( x \in [5, 9] \)

It is evident that the given function \( f(x) \) is not continuous at every integral point.

In particular, \( f(x) \) is not continuous at \( x = 5 \) and \( x = 9 \)

\( \Rightarrow f(x) \) is not continuous in \([5, 9]\).

The differentiability of \( f \) in \((5, 9)\) is checked as follows.

Let \( n \) be an integer such that \( n \in (5, 9) \).

The left hand limit of \( f \) at \( x = n \) is,
\[
\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{[n+h]-[n]}{h} = \lim_{h \to 0} \frac{n-1-n}{h} = \lim_{h \to 0} \frac{-1}{h} = \infty
\]

The right hand limit of \( f \) at \( x = n \) is,
\[
\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{[n+h]-[n]}{h} = \lim_{h \to 0} \frac{n-n}{h} = \lim_{h \to 0} \frac{0}{h} = 0
\]

Since the left and right hand limits of \( f \) at \( x = n \) are not equal, \( f \) is not differentiable at \( x = n \)

\( \therefore f \) is not differentiable in \((5, 9)\).

It is observed that \( f \) does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for \( f(x) = [x] \) for \( x \in [5, 9] \).

(ii) \( f(x) = [x] \) for \( x \in [-2, 2] \)

It is evident that the given function \( f(x) \) is not continuous at every integral point.
In particular, \( f(x) \) is not continuous at \( x = -2 \) and \( x = 2 \)

\[ \Rightarrow f(x) \text{ is not continuous in } [-2, 2]. \]

The differentiability of \( f \) in \((-2, 2)\) is checked as follows.

Let \( n \) be an integer such that \( n \in (-2, 2) \).

The left hand limit of \( f \) at \( x = n \) is,

\[
\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{[n+h] - [n]}{h} = \lim_{h \to 0} \frac{n+1-n}{h} = \lim_{h \to 0} \frac{1}{h} = \infty
\]

The right hand limit of \( f \) at \( x = n \) is,

\[
\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{[n+h] - [n]}{h} = \lim_{h \to 0} \frac{n-n}{h} = \lim_{h \to 0} 0 = 0
\]

Since the left and right hand limits of \( f \) at \( x = n \) are not equal, \( f \) is not differentiable at \( x = n \)

\( \therefore f \) is not differentiable in \((-2, 2)\).

It is observed that \( f \) does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for \( f(x) = [x] \text{ for } x \in [-2, 2] \).

(iii) \( f(x) = x^2 - 1 \text{ for } x \in [1, 2] \)

It is evident that \( f \), being a polynomial function, is continuous in \([1, 2]\) and is differentiable in \((1, 2)\).

It is observed that \( f \) satisfies all the conditions of the hypothesis of Mean Value Theorem.
Hence, Mean Value Theorem is applicable for \( f(x) = x^2 - 1 \) for \( x \in [1, 2] \).

It can be proved as follows.

\[
\begin{align*}
 f(1) &= 1^2 - 1 = 0, \quad f'(2) = 2^2 - 1 = 3 \\
 \therefore \frac{f(b) - f(a)}{b-a} &= \frac{f(2) - f(1)}{2-1} = \frac{3 - 0}{1} = 3 \\
 f'(x) &= 2x \\
 \therefore f'(c) &= 3 \\
 \Rightarrow 2c &= 3 \\
 \Rightarrow c &= \frac{3}{2} = 1.5, \text{ where } 1.5 \in [1, 2]
\end{align*}
\]
Question 1:
\[(3x^2 - 9x + 5)^9\]

Answer

Let \(y = (3x^2 - 9x + 5)^9\)

Using chain rule, we obtain

\[
\frac{dy}{dx} = \frac{d}{dx} (3x^2 - 9x + 5)^9 \\
= 9(3x^2 - 9x + 5)^8 \cdot \frac{d}{dx} (3x^2 - 9x + 5) \\
= 9(3x^2 - 9x + 5)^8 \cdot (6x - 9) \\
= 9(3x^2 - 9x + 5)^8 \cdot 3(2x - 3) \\
= 27(3x^2 - 9x + 5)^8 (2x - 3)
\]

Question 2:
\[\sin^3 x + \cos^6 x\]

Answer

Let \(y = \sin^3 x + \cos^6 x\)

\[
\frac{dy}{dx} = \frac{d}{dx} (\sin^3 x) + \frac{d}{dx} (\cos^6 x) \\
= 3\sin^2 x \cdot \frac{d}{dx} (\sin x) + 6\cos^5 x \cdot \frac{d}{dx} (\cos x) \\
= 3\sin^2 x \cdot \cos x + 6\cos^5 x \cdot (-\sin x) \\
= 3\sin x \cos x (\sin x - 2\cos^4 x)
\]

Question 3:
\[(5x)^{3\cos^2 x}\]

Answer

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Let $y = (5x)^{3 \cos 2x}$

Taking logarithm on both the sides, we obtain

$$\log y = 3 \cos 2x \log 5x$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{y} \frac{dy}{dx} = 3 \left[ \log 5x \cdot \frac{d}{dx} (\cos 2x) + \cos 2x \cdot \frac{d}{dx} (\log 5x) \right]$$

$$\Rightarrow \frac{dy}{dx} = 3y \left[ \log 5x (-\sin 2x) \cdot \frac{d}{dx} (2x) + \cos 2x \cdot \frac{1}{5x} \cdot \frac{d}{dx} (5x) \right]$$

$$\Rightarrow \frac{dy}{dx} = 3y \left[ -2 \sin 2x \log 5x + \frac{\cos 2x}{x} \right]$$

$$\Rightarrow \frac{dy}{dx} = 3y \left[ \frac{3 \cos 2x}{x} - 6 \sin 2x \log 5x \right]$$

$$\therefore \frac{dy}{dx} = (5x)^{3 \cos 2x} \left[ \frac{3 \cos 2x}{x} - 6 \sin 2x \log 5x \right]$$

**Question 4:**

$$\sin^{-1} \left( x \sqrt{x} \right), \ 0 \leq x \leq 1$$

**Answer**

Let $y = \sin^{-1} \left( x \sqrt{x} \right)$

Using chain rule, we obtain
\[
\frac{dy}{dx} = \frac{d}{dx} \sin^{-1}(x\sqrt{x})
\]

\[
= \frac{1}{\sqrt{1-(x\sqrt{x})^2}} \cdot \frac{d}{dx} \left( x\sqrt{x} \right)
\]

\[
= \frac{1}{\sqrt{1-x^3}} \cdot \frac{3}{2} x^{\frac{1}{2}}
\]

\[
= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}
\]

\[
= \frac{3}{2\sqrt{1-x^3}}
\]

**Question 5:**

\[
\cos^{-1}\frac{x}{2\sqrt{2x+7}}, \quad -2 < x < 2
\]

**Answer**
Let \( y = \frac{\arccos x}{\sqrt{2x+7}} \)

By quotient rule, we obtain

\[
\frac{dy}{dx} = \frac{\frac{d}{dx} \left(\frac{\arccos x}{\sqrt{2x+7}}\right)}{\left(\sqrt{2x+7}\right)^2}
\]

\[
= \frac{\sqrt{2x+7} \left(-1 \cdot \frac{d}{dx} \left(\frac{x}{2}\right)\right) - \left(\frac{\arccos x}{2}\right) \frac{1}{2\sqrt{2x+7}} \cdot \frac{d}{dx} (2x+7)}{2x+7}
\]

\[
= \frac{-\sqrt{2x+7} \cdot \frac{1}{2\sqrt{2x+7}} \cdot \frac{d}{dx} (2x+7)}{2x+7}
\]

Question 6:

\[\cot^{-1} \left[ \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right], 0 < x < \frac{\pi}{2}\]

Answer
Let \( y = \cot^{-1}\left[\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}\right] \) \hspace{1cm} ...(1)

Then,
\[
\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}} = \frac{\left(\sqrt{1 + \sin x} + \sqrt{1 - \sin x}\right)^2}{\left(\sqrt{1 + \sin x} - \sqrt{1 - \sin x}\right)^2}
\]
\[
= \frac{(1 + \sin x) + (1 - \sin x) + 2\sqrt{(1 - \sin x)(1 + \sin x)}}{(1 + \sin x) - (1 - \sin x)}
\]
\[
= \frac{2 + 2\sqrt{1 - \sin^2 x}}{2\sin x}
\]
\[
= \frac{1 + \cos x}{\sin x}
\]
\[
= \frac{2\cos^2 \frac{x}{2}}{2\sin \frac{x}{2}\cos \frac{x}{2}}
\]
\[
= \cot \frac{x}{2}
\]

Therefore, equation (1) becomes
\[
y = \cot^{-1}\left(\cot \frac{x}{2}\right)
\]
\[\Rightarrow y = \frac{x}{2}\]
\[\therefore \frac{dy}{dx} = \frac{1}{2} \frac{d}{dx}(x)
\]
\[\Rightarrow \frac{dy}{dx} = \frac{1}{2}
\]

**Question 7:**
\[(\log x)^{\log x}, \ x > 1\]

**Answer**
Let \( y = (\log x)^{\log x} \)
Taking logarithm on both the sides, we obtain
\[ \log y = \log x \cdot \log \left( \log x \right) \]

Differentiating both sides with respect to \( x \), we obtain
\[
\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \left[ \log x \cdot \log \left( \log x \right) \right]
\]
\[
\Rightarrow \frac{1}{y} \frac{dy}{dx} = \log \left( \log x \right) \cdot \frac{d}{dx} \left( \log x \right) + \log x \cdot \frac{d}{dx} \left[ \log \left( \log x \right) \right]
\]
\[
\Rightarrow \frac{dy}{dx} = y \left[ \log \left( \log x \right) \cdot \frac{1}{x} + \log x \cdot \frac{1}{\log x} \cdot \frac{d}{dx} \left( \log x \right) \right]
\]
\[
\Rightarrow \frac{dy}{dx} = y \left[ \frac{1}{x} \log \left( \log x \right) + \frac{1}{x} \right]
\]
\[
\therefore \frac{dy}{dx} = \left( \log x \right)^{\log x} \left[ \frac{1}{x} + \log \left( \log x \right) \right]
\]

Question 8:
\[ \cos \left( a \cos x + b \sin x \right) \], for some constant \( a \) and \( b \).

Answer
Let \( y = \cos \left( a \cos x + b \sin x \right) \)

By using chain rule, we obtain
\[
\frac{dy}{dx} = \frac{d}{dx} \cos \left( a \cos x + b \sin x \right)
\]
\[
\Rightarrow \frac{dy}{dx} = -\sin \left( a \cos x + b \sin x \right) \cdot \frac{d}{dx} \left( a \cos x + b \sin x \right)
\]
\[
= -\sin \left( a \cos x + b \sin x \right) \cdot \left[ a (-\sin x) + b \cos x \right]
\]
\[
= a \sin x - b \cos x \cdot \sin \left( a \cos x + b \sin x \right)
\]

Question 9:
\[ (\sin x - \cos x)^{\sin x - \cos x} \], \( \frac{\pi}{4} < x < \frac{3\pi}{4} \)

Answer
Let\( y = (\sin x - \cos x)^{(\sin x - \cos x)} \)

Taking logarithm on both the sides, we obtain

\[ \log y = \log\left((\sin x - \cos x)^{(\sin x - \cos x)}\right) \]

\[ \Rightarrow \log y = (\sin x - \cos x) \cdot \log(\sin x - \cos x) \]

Differentiating both sides with respect to \( x \), we obtain

\[ \frac{1}{y} \frac{dy}{dx} = \frac{d}{dx}\left[(\sin x - \cos x) \cdot \log(\sin x - \cos x)\right] \]

\[ \Rightarrow \frac{1}{y} \frac{dy}{dx} = \log(\sin x - \cos x) \cdot \frac{d}{dx}(\sin x - \cos x) + (\sin x - \cos x) \cdot \frac{d}{dx}\log(\sin x - \cos x) \]

\[ \Rightarrow \frac{1}{y} \frac{dy}{dx} = \log(\sin x - \cos x) \cdot (\cos x + \sin x) + (\sin x - \cos x) \cdot \frac{1}{(\sin x - \cos x)} \cdot \frac{d}{dx}(\sin x - \cos x) \]

\[ \Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)} \left[ (\cos x + \sin x) \cdot \log(\sin x - \cos x) + (\cos x + \sin x) \right] \]

\[ \therefore \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)} (\cos x + \sin x)[1 + \log(\sin x - \cos x)] \]

**Question 10:**

\( x^a + x^a + a^x + a^a \), for some fixed \( a > 0 \) and \( x > 0 \)

Answer

Let \( y = x^x + x^a + a^x + a^a \)

Also, let \( x^x = u \), \( x^a = v \), \( a^x = w \), and \( a^a = s \)

\[ \therefore y = u + v + w + s \]

\[ \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx} \]

\[ u = x^x \]

\[ \Rightarrow \log u = \log x^x \]

\[ \Rightarrow \log u = x \log x \]

Differentiating both sides with respect to \( x \), we obtain
\[
\frac{1}{u} \frac{du}{dx} = \log x \cdot \frac{d}{dx} \left( x \right) + x \cdot \frac{d}{dx} \left( \log x \right) \\
\Rightarrow \frac{du}{dx} = u \left[ \log x \cdot 1 + x \cdot \frac{1}{x} \right] \\
\Rightarrow \frac{du}{dx} = x^x \left[ \log x + 1 \right] - x^x \left( 1 + \log x \right) \quad \cdots (2) \\
v = x^x \\
\therefore \frac{dv}{dx} = \frac{d}{dx} \left( x^x \right) \\
\Rightarrow \frac{dv}{dx} = ax^{a-1} \quad \cdots (3) \\
w = a^x \\
\Rightarrow \log w = \log a^x \\
\Rightarrow \log w = x \log a \\
Differentiating both sides with respect to \( x \), we obtain \\
\frac{1}{w} \cdot \frac{dw}{dx} = \log a \cdot \frac{d}{dx} \left( x \right) \\
\Rightarrow \frac{dw}{dx} = w \log a \\
\Rightarrow \frac{dw}{dx} = a^x \log a \quad \cdots (4) \\
s = a^x \\
Since \( a \) is constant, \( a^x \) is also a constant. \\
\frac{ds}{dx} = 0 \quad \cdots (5) \\
\therefore \\
From (1), (2), (3), (4), and (5), we obtain \\
\frac{dy}{dx} = x^x \left( 1 + \log x \right) + ax^{a-1} + a^x \log a + 0 \\
= x^x \left( 1 + \log x \right) + ax^{a-1} + a^x \log a \]
Question 11:

\[ (x^2 - 3)^3 + (x - 3)^2 \], for \( x > 3 \)

Answer

Let \( y = (x^2 - 3)^3 + (x - 3)^2 \)

Also, let \( u = x^2 - 3 \) and \( v = (x - 3)^2 \)

\[ \therefore y = u + v \]

Differentiating both sides with respect to \( x \), we obtain

\[
\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(1)}
\]

\[ u = x^2 - 3 \]

\[ \therefore \log u = \log (x^2 - 3) \]

\[ \log u = (x^2 - 3) \log x \]

Differentiating with respect to \( x \), we obtain

\[
\frac{1}{u} \cdot \frac{du}{dx} = \log x \cdot \frac{d}{dx} (x^2 - 3) + (x^2 - 3) \cdot \frac{d}{dx} (\log x)
\]

\[ \Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = \log x \cdot 2x + (x^2 - 3) \cdot \frac{1}{x} \]

\[ \Rightarrow \frac{du}{dx} = x^2 - 3 \cdot \left[ \frac{x^2 - 3}{x} + 2x \log x \right] \]

Also,

\[ v = (x - 3)^2 \]

\[ \therefore \log v = \log (x - 3)^2 \]

\[ \Rightarrow \log v = x^2 \log (x - 3) \]

Differentiating both sides with respect to \( x \), we obtain
\[
\frac{1}{v} \frac{dv}{dx} = \log(x-3) \cdot \frac{d}{dx} \left( \frac{x^2}{x-3} \right) + x^2 \cdot \frac{d}{dx} \left[ \log(x-3) \right]
\]
\[
\Rightarrow \frac{1}{v} \frac{dv}{dx} = \log(x-3) \cdot 2x + x^2 \cdot \frac{1}{x-3} \cdot \frac{d}{dx} (x-3)
\]
\[
\Rightarrow \frac{dv}{dx} = v \left[ 2x \log(x-3) + \frac{x^2}{x-3} \cdot 1 \right]
\]
\[
\Rightarrow \frac{dv}{dx} = (x-3)^{v'} \left[ \frac{x^2}{x-3} + 2x \log(x-3) \right]
\]

Substituting the expressions of \( \frac{du}{dx} \) and \( \frac{dv}{dx} \) in equation (1), we obtain
\[
\frac{dy}{dx} = x^{v-1} \left[ \frac{x^2}{x-3} + 2x \log x \right] + (x-3)^{v'} \left[ \frac{x^2}{x-3} + 2x \log(x-3) \right]
\]

**Question 12:**

\[
\frac{dy}{dx} = 12 \left[ 1 - \cos t \right], x = 10 \left( t - \sin t \right), \quad -\frac{\pi}{2} < t < \frac{\pi}{2}
\]

Find \( \frac{dx}{dt} \), if

**Answer**

It is given that, \( y = 12 \left( 1 - \cos t \right), x = 10 \left( t - \sin t \right) \)

\[
\Rightarrow \frac{dx}{dt} = 10 \cdot \frac{d}{dt} (t - \sin t) = 10 \left( 1 - \cos t \right)
\]
\[
\Rightarrow \frac{dy}{dt} = 12 \cdot \frac{d}{dt} \left[ 1 - \cos t \right] = 12 \cdot \left( 1 - \cos t \right) = 12 \cdot \left[ 0 - (-\sin t) \right] = 12 \sin t
\]
\[
\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12 \sin t}{10 \left( 1 - \cos t \right)} = \frac{12 \cdot 2 \sin \frac{t}{2} \cdot \cos \frac{t}{2}}{2 \sin^2 \frac{t}{2}} = \frac{6 \cot \frac{t}{2}}{2}
\]

**Question 13:**

\[
\frac{dy}{dx} = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}, \quad -1 \leq x \leq 1
\]

Find \( \frac{dx}{dt} \), if

**Answer**
It is given that, \( y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2} \)

\[ \therefore \frac{dy}{dx} = \frac{d}{dx} \left[ \sin^{-1} x + \sin^{-1} \sqrt{1-x^2} \right] \]

\[ \Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left( \sin^{-1} x \right) + \frac{d}{dx} \left( \sin^{-1} \sqrt{1-x^2} \right) \]

\[ \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \cdot \frac{d}{dx} \left( \sqrt{1-x^2} \right) \]

\[ \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx} \left(1-x^2\right) \]

\[ \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} (-2x) \]

\[ \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \]

\[ \therefore \frac{dy}{dx} = 0 \]

Question 14:

If \( x\sqrt{1+y} + y\sqrt{1+x} = 0 \), for, \(-1 < x < 1\), prove that

\[ \frac{dy}{dx} = -\frac{1}{(1+x)^2} \]

Answer

It is given that,

\( x\sqrt{1+y} + y\sqrt{1+x} = 0 \)
⇒ \( x\sqrt{1+y} = -y\sqrt{1+x} \)

Squaring both sides, we obtain
\[
x^2 (1+y) = y^2 (1+x)
\]
⇒ \( x^2 + x^2 y = y^2 + xy^2 \)
⇒ \( x^2 - y^2 = xy^2 - x^2 y \)
⇒ \( x^2 - y^2 = xy(y-x) \)
⇒ \( (x+y)(x-y) = xy(y-x) \)
∴ \( x+y = -xy \)
⇒ \( (1+x) y = -x \)
⇒ \( y = \frac{-x}{(1+x)} \)

Differentiating both sides with respect to \( x \), we obtain
\[
\frac{dy}{dx} = \frac{(1+x) \frac{d}{dx}(x) - x \frac{d}{dx}(1+x)}{(1+x)^2} = - \frac{(1+x) - x}{(1+x)^2} = - \frac{1}{(1+x)^2}
\]

Hence, proved.

**Question 15:**

If \((x-a)^2 + (y-b)^2 = c^2\), for some \( c > 0 \), prove that
\[
\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}} \frac{d^2 y}{dx^2}
\]
is a constant independent of \( a \) and \( b \).

**Answer**

It is given that \((x-a)^2 + (y-b)^2 = c^2\)

Differentiating both sides with respect to \( x \), we obtain
\[
\frac{d}{dx}[(x-a)^2] + \frac{d}{dx}[(y-b)^2] = \frac{d}{dx}(c^2)
\]

\Rightarrow 2(x-a) \cdot \frac{d}{dx}(x-a) + 2(y-b) \cdot \frac{d}{dx}(y-b) = 0

\Rightarrow 2(x-a) \cdot 1 + 2(y-b) \cdot \frac{dy}{dx} = 0

\Rightarrow \frac{dy}{dx} = \frac{-(x-a)}{y-b} \quad \text{...(1)}

\therefore \frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{-(x-a)}{y-b} \right]
Hence, proved.

Question 16:

If \( \cos y = x \cos (a + y) \), with \( \cos a \neq \pm 1 \), prove that \( \frac{dy}{dx} = \frac{\cos^2 (a + y)}{\sin a} \)

Answer
It is given that, \( \cos y = x \cos (a + y) \)

\[
\therefore \frac{d}{dx}[\cos y] = \frac{d}{dx}[x \cos (a + y)]
\]

\[
\Rightarrow -\sin y \frac{dy}{dx} = \cos (a + y) \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}[\cos (a + y)]
\]

\[
\Rightarrow -\sin y \frac{dy}{dx} = \cos (a + y) + x \cdot [-\sin (a + y)] \frac{dy}{dx}
\]

\[
\Rightarrow \left[ x \sin (a + y) - \sin y \right] \frac{dy}{dx} = \cos (a + y)
\]

\[
\text{Since } \cos y = x \cos (a + y), \quad x = \frac{\cos y}{\cos (a + y)}
\]

Then, equation (1) reduces to

\[
\left[ \frac{\cos y}{\cos (a + y)} \cdot \sin (a + y) - \sin y \right] \frac{dy}{dx} = \cos (a + y)
\]

\[
\Rightarrow \left[ \cos y \cdot \sin (a + y) - \sin y \cdot \cos (a + y) \right] \frac{dy}{dx} = \cos^2 (a + y)
\]

\[
\Rightarrow \sin (a + y - y) \frac{dy}{dx} = \cos^2 (a + b)
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{\cos^2 (a + b)}{\sin a}
\]

Hence, proved.

**Question 17:**

If \( x = a(\cos t + t \sin t) \) and \( y = a(\sin t - t \cos t) \), find \( \frac{d^2 y}{dx^2} \)

**Answer**
It is given that, \( x = a(\cos t + t \sin t) \) and \( y = a(\sin t - t \cos t) \)

\[
\frac{dx}{dt} = a \cdot \frac{d}{dt}(\cos t + t \sin t)
\]

\[
= a \left[ -\sin t + t \cdot \frac{d}{dt}(\sin t) + \cos t \cdot \frac{d}{dt}(t) + t \cdot \frac{d}{dt}(\cos t) \right]
\]

\[
= a \left[ -\sin t + t \cdot \cos t \right] = at \cos t
\]

\[
\frac{dy}{dt} = a \cdot \frac{d}{dt}(\sin t - t \cos t)
\]

\[
= a \left[ \cos t - \left( \cos t \cdot \frac{d}{dt}(t) + t \cdot \frac{d}{dt}(\cos t) \right) \right]
\]

\[
= a \left[ \cos t - \{\cos t - t \sin t\} \right] = at \sin t
\]

\[
\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at \sin t}{at \cos t} = \tan t
\]

Then, \[
\frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \tan t \right) = \sec^2 t \cdot \frac{dt}{dx}
\]

\[
= \sec^2 t \cdot \frac{1}{at \cos t} = \frac{\sec^2 t \cdot \frac{dx}{dt}}{at \cos t} = \frac{1}{at \cos t}
\]

\[
= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}
\]

**Question 18:**

If \( f(x) = |x|^3 \), show that \( f^{(n)}(x) \) exists for all real \( x \), and find it.

**Answer**

\[
|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}
\]

It is known that,

Therefore, when \( x \geq 0 \), \( f(x) = |x|^3 = x^3 \)

In this case, \( f'(x) = 3x^2 \) and hence, \( f''(x) = 6x \)

When \( x < 0 \), \( f(x) = |x|^3 = (-x)^3 = -x^3 \)
In this case, \( f'(x) = -3x^2 \) and hence, \( f''(x) = -6x \)

Thus, for \( f(x) = |x|^3 \), \( f''(x) \) exists for all real \( x \) and is given by,

\[
f''(x) = \begin{cases} 
6x, & \text{if } x \geq 0 \\
-6x, & \text{if } x < 0 
\end{cases}
\]

**Question 19:**

Using mathematical induction prove that \( \frac{d}{dx}(x^n) = nx^{n-1} \) for all positive integers \( n \).

**Answer**

To prove: \( P(n): \frac{d}{dx}(x^n) = nx^{n-1} \) for all positive integers \( n \)

For \( n = 1 \),

\[
P(1): \frac{d}{dx}(x) = 1 = 1 \cdot x^{1-1}
\]

\( \therefore P(n) \) is true for \( n = 1 \)

Let \( P(k) \) is true for some positive integer \( k \).

\[
P(k): \frac{d}{dx}(x^k) = kx^{k-1}
\]

That is, \( \frac{d}{dx}(x^k) = kx^{k-1} \)

It has to be proved that \( P(k + 1) \) is also true.

Consider \( \frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k) \)

\[
= x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x^k) \quad \text{[By applying product rule]}
= x \cdot 1 + x \cdot k \cdot x^{k-1}
= x^k + kx^k
= (k+1) \cdot x^k
= (k+1) \cdot x^{(k-1)+1}
\]

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Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for every positive integer $n$.

Hence, proved.

**Question 20:**

Using the fact that $\sin(A + B) = \sin A \cos B + \cos A \sin B$ and the differentiation, obtain the sum formula for cosines.

**Answer**

\[
\sin(A + B) = \sin A \cos B + \cos A \sin B
\]

Differentiating both sides with respect to $x$, we obtain

\[
\frac{d}{dx} \left[ \sin(A + B) \right] = \frac{d}{dx} \left( \sin A \cos B \right) + \frac{d}{dx} \left( \cos A \sin B \right)
\]

\[
\Rightarrow \cos(A + B) \cdot \frac{d}{dx} (A + B) = \cos B \cdot \frac{d}{dx} (\sin A) + \sin A \cdot \frac{d}{dx} (\cos B)
\]

\[
+ \sin B \cdot \frac{d}{dx} (\cos A) + \cos A \cdot \frac{d}{dx} (\sin B)
\]

\[
\Rightarrow \cos(A + B) \cdot \frac{d}{dx} (A + B) = \cos B \cdot \cos A \cdot \frac{dA}{dx} + \sin A \cdot (-\sin B) \cdot \frac{dB}{dx}
\]

\[
+ \sin B \cdot (-\sin A) \cdot \frac{dA}{dx} + \cos A \cos B \cdot \frac{dB}{dx}
\]

\[
\Rightarrow \cos(A + B) \left[ \frac{dA}{dx} + \frac{dB}{dx} \right] = (\cos A \cos B - \sin A \sin B) \cdot \left[ \frac{dA}{dx} + \frac{dB}{dx} \right]
\]

\[
\therefore \cos(A + B) = \cos A \cos B - \sin A \sin B
\]

**Question 22:**

\[
y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}
\]

\[
\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}
\]

If \[
\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix} = 0,
\]

prove that

**Answer**

[Note: The answer is not provided in the document.]
Thus,

\[
y = \begin{vmatrix}
f(x) & g(x) & h(x) \\
l & m & n \\
a & b & c \\
\end{vmatrix}
\]

\[
\Rightarrow y = (mc - nb) f(x) - (lc - na) g(x) + (lb - ma) h(x)
\]

Then,

\[
\frac{dy}{dx} = \frac{d}{dx}[(mc - nb) f(x)] - \frac{d}{dx}[(lc - na) g(x)] + \frac{d}{dx}[(lb - ma) h(x)]
\]

\[
= (mc - nb) f'(x) - (lc - na) g'(x) + (lb - ma) h'(x)
\]

\[
= \begin{vmatrix}
f'(x) & g'(x) & h'(x) \\
l & m & n \\
a & b & c \\
\end{vmatrix}
\]

\[
\frac{dy}{dx} = \begin{vmatrix}
f'(x) & g'(x) & h'(x) \\
l & m & n \\
a & b & c \\
\end{vmatrix}
\]

Thus,

**Question 23:**

If \(y = e^{\cos^{-1}x}, \quad -1 \leq x \leq 1\), show that

\[
\left(1 - x^2\right) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0
\]

**Answer**

It is given that, \(y = e^{\cos^{-1}x}\)
Taking logarithm on both the sides, we obtain
\[ \log y = a \cos^{-1} x \log e \]
\[ \log y = a \cos^{-1} x \]

Differentiating both sides with respect to \( x \), we obtain
\[ \frac{1}{y} \frac{dy}{dx} = a \times \frac{-1}{\sqrt{1-x^2}} \]
\[ \Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}} \]

By squaring both the sides, we obtain
\[ \left( \frac{dy}{dx} \right)^2 = \frac{a^2 y^2}{1-x^2} \]
\[ \Rightarrow (1-x^2) \left( \frac{dy}{dx} \right)^2 = a^2 y^2 \]
\[ (1-x^2) \left( \frac{dy}{dx} \right)^2 = a^2 y^2 \]

Again differentiating both sides with respect to \( x \), we obtain
\[ \left( \frac{dy}{dx} \right)^2 \frac{d}{dx} (1-x^2) + (1-x^2) \frac{d}{dx} \left( \frac{dy}{dx} \right)^2 = a^2 \frac{dy}{dx} \left( y^2 \right) \]
\[ \Rightarrow \left( \frac{dy}{dx} \right)^2 (-2x) + (1-x^2) \times 2 \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2} = a^2 \cdot 2y \cdot \frac{dy}{dx} \]
\[ \Rightarrow \left( \frac{dy}{dx} \right)^2 (-2x) + (1-x^2) \times 2 \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2} = a^2 \cdot 2y \cdot \frac{dy}{dx} \]
\[ \Rightarrow -x \frac{dy}{dx} + (1-x^2) \frac{d^2 y}{dx^2} = a^2 . y \quad \left[ \frac{dy}{dx} \neq 0 \right] \]
\[ \Rightarrow (1-x^2) \frac{d^3 y}{dx^3} - x \frac{dy}{dx} - a^2 y = 0 \]
Hence, proved.