Mathematics Notes for Class 12 chapter 7. Integrals

Let f(x) be a function. Then, the collection of all its primitives is called the **indefinite integral** of f(x) and is denoted by $\int f(x)dx$. Integration as inverse operation of differentiation. If $d/dx\ \{\varphi(x)) = f(x)$, $\int f(x)dx = \varphi(x) + C$, where C is called the constant of integration or arbitrary constant.

Symbols $f(x) \rightarrow$ Integrand
$f(x)dx \rightarrow$ Element of integration
$\int \rightarrow$ Sign of integral
$\varphi(x) \rightarrow$ Anti-derivative or primitive or integral of function f(x)
The process of finding functions whose derivative is given, is called anti-differentiation or integration.

1. $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1 \Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

2. $\frac{d}{dx}(\log_e x) = \frac{1}{x} \Rightarrow \int \frac{1}{x} dx = \log_e |x| + C$

3. $\frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x dx = e^x + C$

4. $\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x, a > 0, a \neq 1 \Rightarrow \int a^x dx = \frac{a^x}{\log_e a} + C$

5. $\frac{d}{dx}(-\cos x) = \sin x \Rightarrow \int \sin x\, dx = -\cos x + C$

6. $\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x\, dx = \sin x + C$

7. $\frac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x\, dx = \tan x + C$

8. $\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x \Rightarrow \int \operatorname{cosec}^2 x\, dx = -\cot x + C$

9. $\frac{d}{dx}(\sec x) = \sec x \tan x \Rightarrow \int \sec x \tan x\, dx = \sec x + C$

10. $\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x \Rightarrow \int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$

11. $\frac{d}{dx}(\log \sin x) = \cot x \Rightarrow \int \cot x\, dx = \log|\sin x| + C$

12. $\frac{d}{dx}(-\log \cos x) = \tan x \Rightarrow \int \tan x\, dx = -\log|\cos x| + C$

13. $\frac{d}{dx}[\log(\sec x + \tan x)] = \sec x \Rightarrow \int \sec x\, dx = \log|\sec x + \tan x| + C$

14. $\frac{d}{dx}[\log(\operatorname{cosec} x - \cot x)] = \operatorname{cosec} x$

$\Rightarrow \int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$

15. $\frac{d}{dx} \sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{x\sqrt{a^2 - x^2}} \Rightarrow \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

16. $\frac{d}{dx} \cos^{-1}\left(\frac{x}{a}\right) = \int \frac{-1}{\sqrt{a^2 - x^2}} \Rightarrow \int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + C$

17. $\frac{d}{dx}\left(\frac{1}{a} \tan^{-1} \frac{x}{a}\right) = \frac{1}{a^2 + x^2} \Rightarrow \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

18. $\frac{d}{dx}\left(\frac{1}{a} \cot^{-1} \frac{x}{a}\right) = \frac{-1}{a^2 + x^2} \Rightarrow \int \frac{-1}{a^2 + x^2} dx \frac{1}{a} = \cot^{-1}\left(\frac{x}{a}\right) + C$

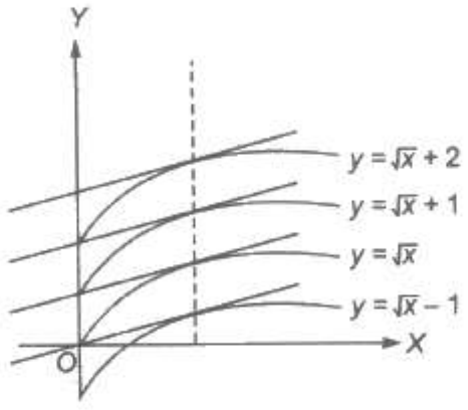
19. $\frac{d}{dx}\left(\frac{1}{a} \sec^{-1} \frac{x}{a}\right) = \frac{1}{x\sqrt{x^2 - a^2}} \Rightarrow \int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + C$

20. $\frac{d}{dx}\left(\frac{1}{a} \operatorname{cosec}^{-1} \frac{x}{a}\right) = \frac{-1}{x\sqrt{x^2 - a^2}}$

$\Rightarrow \qquad \int \frac{-1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + C$

Geometrical Interpretation of Indefinite Integral

If d/dx {φ(x)} = f (x), then ∫f(x)dx = φ(x) + C. For different values of C, we get different functions, differing only by a constant. The graphs of these functions give us an infinite family of curves such that at the points on these curves with the same x-coordinate, the tangents are parallel as they have the same slope φ'(x) = f(x).



Consider the integral of 1/2√x
i.e., ∫1/2√xdx = √x + C, C ∈ R

Above figure shows some members of the family of curves given by y = + C for different C ∈ R.

Comparison between Differentiation and Integration

(i) Both differentiation and integration are linear operator on functions as

d/dx {af(x) ± bg(x)} = a d/dx{f(x) ± d/dx{g(x)}
and ∫[a.f(x) ± b.g(x)dx = a ∫f(x)dx ± b ∫g(x)dx

(ii) All functions are not differentiable, similarly there are some function which are not integrable.

(iii) Integral of a function is always discussed in an interval but derivative of a function can be discussed in a interval as well as on a point.

(iv) Geometrically derivative of a function represents slope of the tangent to the graph of function at the point. On the other hand, integral of a function represents an infinite family of curves placed parallel to each other having parallel tangents at points of intersection of the curves with a line parallel to Y-axis.

Rules of Integration

(i) $\frac{d}{dx}\{\int f(x)dx\} = f(x)$

(ii) $\int k \cdot f(x)dx = k\int f(x)dx$

(iii) $\int \{f_1(x) \pm f_2(x) \pm f_3(x) \pm \ldots \pm f_n(x)\}dx$

$= \int f_1(x)\, dx \pm \int f_2(x)\, dx \pm \int f_3(x)\, dx \pm \ldots \pm \int f_n(x)\, dx$

(iv) $\int f'\{g(x)\} \cdot g'(x)dx = f\{g(x)\} + C$

Method of Substitution

(i) $\frac{d}{dx}\{\int f(x)dx\} = f(x)$

(ii) $\int k \cdot f(x)dx = k\int f(x)dx$

(iii) $\int \{f_1(x) \pm f_2(x) \pm f_3(x) \pm \ldots \pm f_n(x)\}dx$

$= \int f_1(x)\, dx \pm \int f_2(x)\, dx \pm \int f_3(x)\, dx \pm \ldots \pm \int f_n(x)\, dx$

(iv) $\int f'\{g(x)\} \cdot g'(x)dx = f\{g(x)\} + C$

Basic Formulae Using Method of Substitution

If $\int f(x)\,dx = \phi(x)$, then $\int f(ax+b)dx = \frac{1}{a}\phi(ax+b)$.

1. $\int (ax+b)^n\,dx = \frac{1}{a}\cdot\frac{(ax+b)^{n+1}}{n+1} + C, n \neq -1$
2. $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log|ax+b| + C$
3. $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$
4. $\int a^{bx+c}\,dx = \frac{1}{b}\cdot\frac{a^{bx+c}}{\log a} + C, a > 0$ and $a \neq 1$
5. $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$
6. $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$
7. $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + C$
8. $\int \operatorname{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + C$
9. $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + C$
10. $\int \operatorname{cosec}(ax+b)\cot(ax+b)dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + C$
11. $\int \tan(ax+b)\,dx = -\frac{1}{a}\log|\cos(ax+b)| + C$
12. $\int \cot(ax+b)\,dx = \frac{1}{a}\log|\sin(ax+b)| + C$
13. $\int \sec(ax+b)\,dx = \frac{1}{a}\log|\sec(ax+b) + \tan(ax+b)| + C$
14. $\int \operatorname{cosec}(ax+b)\,dx = \frac{1}{a}\log|\operatorname{cosec}(ax+b) - \cot(ax+b)| + C$
15. $\int \tan x\,dx = \log|\sec x| + C$
16. $\int \cot x\,dx = -\log|\operatorname{cosec} x| + C$
17. $\int \sec x\,dx = \log|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)| + C$
18. $\int \operatorname{coses} x\,dx = \log|\tan\frac{x}{2}| + C$

If degree of the numerator of the integrand is equal to or greater than that of denominator divide the numerator by the denominator until the degree of the remainder is less than that of denominator i.e., (Numerator / Denominator) = Quotient + (Remainder / Denominator)

Trigonometric Identities Used for Conversion of Integrals into the Integrable Forms

(i) $\sin^2 nx = \frac{1 - \cos 2nx}{2}$

(ii) $\cos^2 nx = \frac{1 + \cos 2nx}{2}$

(iii) $\sin nx = 2 \sin \frac{nx}{2} \cos \frac{nx}{2}$

(iv) $\sin^3 nx = \frac{3}{4} \sin nx - \frac{1}{4} \sin 3nx$

(v) $\cos^3 nx = \frac{3}{4} \cos nx + \frac{1}{4} \cos 3nx$

(vi) $\tan^2 nx = \sec^2 nx - 1$

(vii) $\cot^2 nx = \operatorname{cosec}^2 nx - 1$

(viii) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$

$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$

$2 \sin A \sin B = \cos (A - B) - \cos (A + B)$

Standard Substitution

S. No.	Function	Substitution
(i)	$(a^2 + x^2), \sqrt{x^2 + a^2}, \frac{1}{\sqrt{x^2 + a^2}}$	$x = a\tan\theta$ or $a\cot\theta$ or $a\sinh\theta$
(ii)	$(a^2 - x^2), \sqrt{a^2 - x^2}, \frac{1}{\sqrt{a^2 - x^2}}$	$x = a\sin\theta$ or $a\cos\theta$
(iii)	$(x \pm \sqrt{x^2 \pm a^2})^n$	expression indside the bracket $= t$
(iv)	$\frac{2x}{a^2 - x^2}, \frac{2x}{a^2 + x^2}, \frac{a^2 - x^2}{a^2 + x^2}$	$x = a\tan\theta$
(v)	$\frac{1}{(x + a)^{1-\frac{1}{n}}(x + b)^{1+\frac{1}{n}}}$ $(n \in N, n > 1)$	$\frac{x + a}{x + b} = t$
(vi)	$(x^2 - a^2), \sqrt{x^2 - a^2}, \frac{1}{\sqrt{x^2 - a^2}}$	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$ or $a\cosh\theta$
(vii)	$\sqrt{\frac{a - x}{a + x}}$ or $\sqrt{\frac{a + x}{a - x}}$	$x = a\cos 2\theta$
(viii)	$\sqrt{\frac{x - \alpha}{\beta - x}}$ or $\sqrt{(x - \alpha)(\beta - x)}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$
(ix)	$\sqrt{2ax - x^2}$	$x = a(1 - \cos\theta)$
(x)	$\sqrt{\frac{x}{a + x}}, \sqrt{\frac{a + x}{x}}, \sqrt{x(a + x)},$	$x = a\tan^2\theta$ or $a\cot^2\theta$
(xi)	$\sqrt{\frac{x}{a - x}}; \sqrt{\frac{a - x}{x}}, \sqrt{x(a - x)}, \frac{1}{\sqrt{x(a - x)}}$	$x = a\sin^2\theta$ or $a\cos^2\theta$
(xii)	$\sqrt{\frac{x}{x - a}}; \sqrt{\frac{x - a}{x}}, \sqrt{x(x - a)}, \frac{1}{\sqrt{x(x - a)}}$	$x = a\sec^2\theta$ or $a\,\text{cosec}^2\,\theta$

Special Integrals

(i) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$

(ii) $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log\left|\frac{a + x}{a - x}\right| + C = \frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right) + C$

(iii) $\int \frac{dx}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x - a}{x + a}\right| + C = -\frac{1}{a} \coth^{-1}\left(\frac{x}{a}\right) + C$

(iv) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$

(v) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log|x + \sqrt{x^2 - a^2}| + C = \cosh^{-1}\left(\frac{x}{a}\right) + C$

(vi) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log|x + \sqrt{x^2 + a^2}| + C = \sinh^{-1}\left(\frac{x}{a}\right) + C$

(vii) $\int \sqrt{x^2 + a^2}\, dx = \frac{1}{2}\left[x\sqrt{x^2 + a^2} + a^2 \log|x + \sqrt{x^2 + a^2}|\right] + C$

(viii) $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right] + C$

(ix) $\int \sqrt{x^2 - a^2} dx = \frac{1}{2}\left[x\sqrt{x^2 - a^2} - a^2 \log|x + \sqrt{x^2 - a^2}|\right] + C$

(x) $\int (px + q)\sqrt{ax^2 + bx + c}\, dx = \frac{p}{2a} \int (2ax + b)\sqrt{ax^2 + bx + c}\, dx$

$+\left(\frac{q - pb}{2a}\right) \int \sqrt{ax^2 + bx + c}\, dx$

Important Forms to be converted into Special Integrals

(i) Form I

$$\int \frac{1}{ax^2 + bx + c} dx \text{ or } \int \frac{1}{\sqrt{ax^2 + bx + c}} dx$$

Express $ax^2 + bx + c$ as sum or difference of two squares *i.e.*,

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

(ii) Form II

$$\int \frac{px+q}{ax^2+bx+c}\,dx \text{ or } \int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$$

Put px + q = λd / dx (ax2 + bx + c) + mu;

Now, find values of λ and mu; and integrate.

(iii) Form III

$$\int \frac{P(x)}{ax^2+bx+c}\,dx,$$

when P(x) is a polynomial of degree 2 or more carry out the dimension and express in the form $P(x) / (ax^2 + bx + c) = Q(x) + R(x) / (ax^2 + bx + c)$, where R(x) is a linear expression or constant, then integral reduces to the form discussed earlier.

(iv) Form IV

$$\int \frac{x^2+a^2}{x^4+kx^2+a^4}\,dx \text{ or } \int \frac{x^2-a^2}{x^4+kx^2+a^4}\,dx$$

After dividing both numerator and denominator by x^2, put $x - a^2 / x = t$ or $x + (a^2 / x) = t$.

(v) Form V

$$\int \frac{dx}{a+b\sin^2 x}, \int \frac{dx}{a+b\cos^2 x}, \int \frac{dx}{a\sin^2 x+b\cos^2 x},$$

$$\int \frac{dx}{a\sin^2 x+b\cos^2 x+c}, \int \frac{dx}{(a\sin x+b\cos x)^2}$$

To evaluate the above type of integrals, we proceed as follow
(a) Divide numerator and denominator by $\cos^2 x$.
(b) Rreplace $\sec^2 x$, if any in denominator by $1 + \tan^2 x$.
(c) Put tan x = t, then $\sec^2 x dx = dt$

(vi) Form VI

$$\int \frac{dx}{a+b\sin x}, \int \frac{dx}{a+b\cos x}, \quad \int \frac{dx}{a\sin x+b\cos x}, \quad \int \frac{dx}{a\sin x+b\cos x+c}$$

To evaluate the above type of integrals, we proceed as follow

(a) Put $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$, then $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

(b) Replace $1+\tan^2\frac{x}{2}$ by $\sec^2\frac{x}{2}$.

(c) Put $\tan\frac{x}{2} = t$, $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$ i.e., $\sec^2\frac{x}{2}\,dx = 2\,dt$

(vii) Form VII

$$\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx,$$

Write numerator

$= \lambda$ (differentiation of denominator) $+\mu$ (denominator)

i.e., $a\sin x + b\cos x = \lambda\,(c\cos x - d\sin x) + \mu(c\sin x + d\cos x)$

$$\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx = \lambda \int \frac{c\cos x - d\sin x}{c\sin x + d\cos x}\,dx + \mu \int \frac{c\sin x + d\cos x}{c\sin x + d\cos x}\,dx$$

$$= \lambda \log|c\sin x + d\cos x| + \mu x + C$$

(viii) Form VIII

$$\int \frac{a \sin x + b \cos x + c}{p \sin x + q \cos x + r}\,dx$$

Write numerator

= λ (differentiation of denominator) + μ(denominator) + γ

i.e., $a \sin x + b \cos x + c = \lambda(p \cos x - q \sin x)$

$$+ \mu(p \sin x + q \cos x + r) + \gamma$$

$$\int \frac{a \sin x + b \cos x + c}{p \sin x + q \cos x + r}\,dx = \lambda \int \frac{p \cos x - q \sin x}{p \sin x + q \cos x + r}\,dx\ x$$

$$+ \mu \int \frac{p \sin x + q \cos x + r}{p \sin x + q \cos x + r}\,dx + \gamma \int \frac{1}{p \sin x + q \cos x + r}\,dx$$

$$= \lambda \log|\, p \sin x + q \cos x + r|$$

$$+ \mu x + \gamma \int \frac{1}{p \sin x + q \cos x + r}\,dx$$

(ix) Form IX

$$\int \frac{x^2+1}{x^4+\lambda x^2+1}\,dx,\ \int \frac{x^2-1}{x^4+\lambda x^2+1}\,dx,\ \int \frac{1}{x^4+\lambda x^2+1}\,dx$$

To evaluate the above type of integrals, we proceed as follows

- Divide numerator and denominator by x^2
- Express the denominator of integrands in the form of $(x + 1/x)^2 \pm k^2$
- Introduce $(x + 1/x)$ or $d(x - 1/x)$ or both in numerator.
- Put $x + 1/x = t$ or $x - 1/x = t$ as the case may be.
- Integral reduced to the form of $\int 1 / x^2 + a^2 dx$ or $\int 1 / x^2 + a^2 dx$

(x) Form X

$$\int \sqrt{\tan x}\,dx, \int \sqrt{\cot x}\,dx, \int \frac{dx}{\sin^4 x + \cos^4 x}, \int \frac{dx}{\sin^6 x + \cos^6 x},$$

$$\int \frac{\pm \sin x \pm \cos x}{a + b \sin x \cdot \cos x}\,dx,$$

Put $\tan x = t^2 \Rightarrow d(\tan x) = d(t^2) \Rightarrow \sec^2 x = 2t\,dt$

$$\Rightarrow \int \frac{2t^2}{t^4+1}\,dt = \int \frac{t^2+1}{t^4+1}\,dt + \int \frac{t^2-1}{t^4+1}\,dt$$, now same as Form (ix).

Integration by Parts

This method is used to integrate the product of two functions. If f(x) and g(x) be two integrable functions, then

$$\int \underset{\text{I}}{f(x)} \cdot \underset{\text{II}}{g(x)}\, dx = f(x)\int g(x)\, dx - \int \left\{\frac{d}{dx} f(x) \int g(x)\, dx\right\} dx$$

(i) We use the following preferential order for taking the first function. Inverse→ Logarithm→ Algebraic → Trigonometric→ Exponential. In short we write it HATE.

(ii) If one of the function is not directly integrable, then we take it a the first function.

(iii) If only one function is there, i.e., ∫log x dx, then 1 (unity) is taken as second function.

(iv) If both the functions are directly integrable, then the first function is chosen in such a way that its derivative vanishes easily or the function obtained in integral sign is easilY integrable.

Integral of the Form

$$\int \mathbf{e^x \{f(x) + f'(x)\}\, dx}$$

$$\int e^x\{f(x) + f'(x)\}dx = \int \underset{\text{I}}{e^x} f(x)dx + \int \underset{\text{II}}{e^x} f'(x)dx$$

$$= f(x)\int e^x dx - \int \{f'(x)\int e^x dx\}dx + \int e^x f'(x)dx$$

$$= f(x)e^x - \int f'(x)e^x dx + \int e^x f'(x)dx = f(x)e^x$$

$$\int e^{ax} \sin(bx + c)\, dx = \frac{e^{ax}}{a^2 + b^2}\{a \sin(bx + c) - b\cos(bx + c)\} + k$$

$$\int e^{ax} \cos(bx + c)dx = \frac{e^{ax}}{a^2 + b^2}\{a \cos(bx + c) + b\sin(bx + c)\} + k$$

Integration Using Partial Fractions

(i) If f(x) and g(x) are two polynomials, then f(x) / g(x) defines a rational algebraic function of x. If degree of f(x) < degree of g(x), then f(x) / g(x) is called a proper rational function.

(ii) If degree of f(x) ≥ degree of g(x), then f(x) /g(x) is called an improper g(x) rational function.

(iii) If f(x) / g(x) isan improper rational function, then we divide f(x) by g(x) g(x) and convert it into a proper rational function as f(x) / g(x) = φ(x) + h(x) / g(x).

(iv) Any proper rational function f(x) / g(x) can be expressed as the sum of rational functions each having a simple factor of g(x). Each such fraction is called a partial fraction and the process of obtaining them, is called the resolution or decomposition of f(x) /g(x) partial fraction.

S. No.	Type of proper rational function	Partial fraction
(i)	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
(ii)	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}, a \neq b \neq c$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
(iii)	$\frac{px+q}{(x-a)^3}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}$
(iv)	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$
(v)	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$, where x^2+bx+c cannot be factorised.	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$
(vi)	$\frac{px^3+qx^2+rx+s}{(x^2+ax+b)(x^2+cx+d)}$, where (x^2+ax+b) and (x^2+cx+d) can not be factorised.	$\frac{Ax+B}{x^2+ax+b}+\frac{Cx+D}{x^2+cx+d}$

Shortcut for Finding Values of A, B and C etc.

Case I. When g(x) is expressible as the product of non-repeated line factors.

Let $g(x)=(x-a_1)(x-a_2)(x-a_3)\dots(x-a_n)$,

then $$\frac{f(x)}{g(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\frac{A_3}{x-a_3}+\dots+\frac{A_n}{x-a_n}$$

Now, $$A_1=\frac{f(a_1)}{(a_1-a_2)(a_1-a_3)(a_1-a_4)\dots(a_1-a_n)}$$

$$A_2=\frac{f(a_2)}{(a_2-a_1)(a_2-a_3)(a_2-a_4)\dots(a_2-a_n)}\dots$$

$$A_n=\frac{f(a_n)}{(a_n-a_1)(a_n-a_2)(a_n-a_3)\dots(a_n-a_{n-1})}$$

Trick To find A^p put x = a in numerator and denominator after P deleting the factor ($x — a_p$).

Case II. When g(x) is expressible as product of repeated linear factors.

Let $g(x)=(x-a)^k(x-a_1)(x-a_2)\dots(x-a_n)$,

then $$\frac{f(x)}{g(x)}=\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\dots+\frac{A_k}{(x-a)^k}+\frac{B_1}{(x-a_1)}+\frac{B_2}{(x-a_2)}+\dots+\frac{B_n}{(x-a)^n}$$

Here, all the constant cannot be calculated by using the method in Case I. However, B_l, B_2, B_3, … , B_n can be found using the same method i.e., shortcut can be applied only in the case of non-repeated linear factor.

Integration of Irrational Algebraic Function

Irrational function of the form of $(ax + b)^{1/n}$ and x can be evaluated by substitution $(ax + b) = t^n$, thus

$$\int f\{x,(ax+b)^{1/n}\}dx = \int f\left(\frac{t^n-b}{a},t\right)\frac{nt^{n-1}}{a}dt.$$

(i) $\int \frac{dx}{(Ax+B)\sqrt{Cx+D}}$, substitute $Cx+D=t^2$, then the substitution will reduce the given integral into $\int \frac{2dt}{At^2-AD+BC}$.

(ii) $\int \frac{dx}{Ax^2+B\sqrt{Cx+D}}$, substitute $Cx+D=t^2$, then the substitution will reduce the given integral into $\int \frac{2C\,dt}{At^4-2DAt^2+(AD^2+BC^2)}$.

(iii) $\int \frac{dx}{(x-k)^r\sqrt{ax^2+bx+c}}$, substitute $x-k=\frac{1}{t}$, then the substitution will reduce the given integral into $\int \frac{t^{r-1}}{At^2+Bt+c}dt$.

(iv) $\int \frac{1}{(Ax^2+B)\sqrt{Cx^2+D}}dx$, substitute $x=\frac{1}{t}$, there

$$\int \frac{1}{(Ax^2+B)\sqrt{Cx^2+D}} = \int \frac{\left(-\frac{1}{t^2}\right)dt}{\left(\frac{A}{t^2}+B\right)\sqrt{\frac{C}{t^2}+D}}$$

$$= -\int \frac{t}{(A+Bt^2)\sqrt{C+Dt^2}}dt.$$

Now, put $C+Dt^2=u^2$ reduces it into the form $\int \frac{1}{u^2\pm a^2}du$,

$$\int \frac{ax^2+bx+c}{(dx+e)\sqrt{fx^2+gx+h}}dx$$

Here, we write

$$ax^2+bx+c = A_1(dx+e)\frac{d}{dx}(fx^2+gx+h)+B_1(dx+e)+C_1$$

where, A_1, B_1 and C_1 are constants.

Integrals of the Type $(bx^m + bx^n)^P$

Case I. If P ∈ N (natural number) we expand the binomial theorem and integrate.

Case II. If P ∈ Z (integers), put $x = p^k$, where k denominator of m and n.

Case III. If (m+1)/n is an integer, we put $(a + bx^n) = r^k$, where k is th denominator of the fraction.

Integration of Hyperbolic Functions

- $\int \sinh x\, dx = \cosh x + C$
- $\int \cosh x\, dx = \sinh x + C$
- $\int \text{sech}^2 x\, dx = \tanh x + C$
- $\int \text{cosech}^2 x\, dx = -\coth x + C$
- $\int \text{sech}\, x \tanh x\, dx = -\text{sech}\, x + C$
- $\int \text{cosech}\, x \coth x\, dx = -\text{cosech}\, x + C$

Case IV If $\{(m+1)/n\} + P$ is an integer, we put $(a + bx^n) = r^k x^n$ is the denominator of the fraction p.

Important Points to be Remembered

(i) (a) Anti-derivative of signum exists in that interval in which $x=0$ is not included.

(b) Anti-derivative of odd function is always even and of even function is always odd.

(ii) If $I_n=\int x^n e^{ax}dx$, then $I_n=\dfrac{x^n e^{ax}}{a}-\dfrac{n}{a}I_{n-1}$

(iii) (a) $\int(\log x)dx = x\log x - x$

(b) $\displaystyle\int\frac{1}{\log x}dx = \log(\log x)+\log x+\frac{(\log x)^2}{2(2!)}+\frac{(\log x)^3}{3(3!)}+\dots$

(iv) $\displaystyle\int\frac{a\cos x+b\sin x}{c\cos x+d\sin x}dx=\frac{ac+bd}{c^2+d^2}x+\frac{ad-bc}{c^2+d^2}\log|c\cos x+d\sin x|+k$

(v) $\displaystyle\int\frac{\sin^n x}{\cos^m x}dx=\frac{1}{m-1}\cdot\frac{\sin^{n-1}x}{\cos^{m-1}x}-\frac{n-1}{m-1}\int\frac{\sin^{n-2}x}{\cos^{m-2}x}dx$

(vi) (a) $\displaystyle\int a^x\cos(bx+c)\,dx=\frac{a^x}{(\log a)^2+b^2}[(\log a)\cos(bx+c)+b\sin(bx+c)]+k$

(b) $\displaystyle\int a^x\sin(bx+c)\,dx=\frac{a^x}{(\log a)^2+b^2}[(\log a)\sin(bx+c)-b\cos(bx+c)]+k$

(vii) (a) $\displaystyle\int xe^{ax}\cos(bx+c)dx=\frac{xe^{ax}}{a^2+b^2}[a\cos(bx+c)+b\sin(bx+c)]$

$$-\frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\cos(bx+c)+2ab\sin(bx+c)]+k$$

(b) $\displaystyle\int xe^{ax}\sin(bx+c)dx=\frac{xe^{ax}}{a^2+b^2}[a\sin(bx+c)-b\cos(bx+c)]$

$$-\frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\sin(bx+c)-2ab\cos(bx+c)]+k$$

(viii) (a) $\displaystyle\sin^n x\,dx=\frac{-\cos x\cdot\sin^{n-1}x}{n}+\frac{n-1}{n}\int\sin^{n-2}x\,dx$

(b) $\displaystyle\int\cos^n x\,dx=\frac{\sin x\cdot\cos^{n-1}x}{n}+\frac{n-1}{n}\int\cos^{n-2}x\,dx$

(c) $\displaystyle\int\tan^n x\,dx=\frac{\tan^{n-1}x}{n-1}-\int\tan^{n-2}x\,dx$