Mathematics Notes for Class 12 chapter 5. Continuity and Differentiability

Derivative

The rate of change of a quantity $y$ with respect to another quantity $x$ is called the derivative or differential coefficient of $y$ with respect to $x$.

Differentiation of a Function

Let $f(x)$ is a function differentiable in an interval $[a, b]$. That is, at every point of the interval, the derivative of the function exists finitely and is unique. Hence, we may define a new function $g: [a, b] \rightarrow \mathbb{R}$, such that, $\forall \ x \in [a, b]$, $g(x) = f'(x)$.

This new function is said to be differentiation (differential coefficient) of the function $f(x)$ with respect to $x$ and it is denoted by $df(x) / dx$ or $Df(x)$ or $f'(x)$.

Differentiation ‘from First Principle

Let $f(x)$ is a function finitely differentiable at every point on the real number line. Then, its derivative is given by

$$f'(x) = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Standard Differentiations

1. $\frac{d}{dx} (x^n) = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{R}$
2. $\frac{d}{dx} (k) = 0$, where $k$ is constant.
3. $\frac{d}{dx} (e^x) = e^x$
4. $\frac{d}{dx} (a^x) = a^x \log a$, $a > 0$, $a \neq 1$
5. \( \frac{d}{dx} (\log_e x) = \frac{1}{x}, x > 0 \)

6. \( \frac{d}{dx} (\log_a x) = \frac{1}{x} \left( \log_a e \right) = \frac{1}{x \log_e a} \)

7. \( \frac{d}{dx} (\sin x) = \cos x \)

8. \( \frac{d}{dx} (\cos x) = -\sin x \)

9. \( \frac{d}{dx} (\tan x) = \sec^2 x, x \neq (2n + 1) \frac{\pi}{2}, n \in \mathbb{I} \)

10. \( \frac{d}{dx} (\cot x) = -\csc^2 x, x \neq n\pi, n \in \mathbb{I} \)

11. \( \frac{d}{dx} (\sec x) = \sec x \tan x, x \neq (2n + 1) \frac{\pi}{2}, n \in \mathbb{I} \)

12. \( \frac{d}{dx} (\csc x) = -\csc x \cot x, x \neq n\pi, n \in \mathbb{I} \)

13. \( \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1 \)

14. \( \frac{d}{dx} (\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1 \)

15. \( \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2} \)

16. \( \frac{d}{dx} (\cot^{-1} x) = -\frac{1}{1+x^2} \)

17. \( \frac{d}{dx} (\sec^{-1} x) = \frac{1}{|x| \sqrt{x^2-1}}, |x| > 1 \)

18. \( \frac{d}{dx} (\csc^{-1} x) = -\frac{1}{|x| \sqrt{x^2-1}}, |x| > 1 \)

19. \( \frac{d}{dx} (\sinh x) = \cosh x \)

20. \( \frac{d}{dx} (\cosh x) = \sinh x \)

21. \( \frac{d}{dx} (\tanh x) = \text{sech}^2 x \)

**Fundamental Rules for Differentiation**
(i) \( \frac{d}{dx}(cf(x)) = c \frac{d}{dx} f(x) \), where \( c \) is a constant.

(ii) \( \frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x) \) (sum and difference rule)

(iii) \( \frac{d}{dx}(f(x)g(x)) = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x) \) (product rule)

Geometrically Meaning of Derivative at a Point

Geometrically derivative of a function at a point \( x = c \) is the slope of the tangent to the curve \( y = f(x) \) at the point \( (c, f(c)) \).

Slope of tangent at \( P = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \left( \frac{df(x)}{dx} \right)_{x = c} \) or \( f'(c) \).

Differentiation of a constant function is zero i.e., \( \frac{d}{dx} (c) = 0 \).

(v) if \( \frac{d}{dx} f(x) = \phi(x) \), then \( \frac{d}{dx} f(ax + b) = a \phi(ax + b) \)

(vi) Differentiation of a constant function is zero i.e., \( \frac{d}{dx} (c) = 0 \).

Different Types of Differentiable Function

1. Differentiation of Composite Function (Chain Rule)

If \( f \) and \( g \) are differentiable functions in their domain, then \( fog \) is also differentiable and

\( (fog)'(x) = f'(g(x)) \cdot g'(x) \)

More easily, if \( y = f(u) \) and \( u = g(x) \), then \( \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \).

If \( y \) is a function of \( u \), \( u \) is a function of \( v \) and \( v \) is a function of \( x \). Then,

\( \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} \).

2. Differentiation Using Substitution
In order to find differential coefficients of complicated expression involving inverse trigonometric functions some substitutions are very helpful, which are listed below.

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Function</th>
<th>Substitution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>$\sqrt{a^2 - x^2}$</td>
<td>$x = a \sin \theta$ or $a \cos \theta$</td>
</tr>
<tr>
<td>(ii)</td>
<td>$\sqrt{a^2 + x^2}$</td>
<td>$x = a \tan \theta$ or $a \cot \theta$</td>
</tr>
<tr>
<td>(iii)</td>
<td>$\sqrt{x^2 - a^2}$</td>
<td>$x = a \sec \theta$ or $a \cosec \theta$</td>
</tr>
<tr>
<td>(iv)</td>
<td>$\sqrt{a + x}$ and $\sqrt{a - x}$</td>
<td>$x = a \cos \theta$</td>
</tr>
<tr>
<td>(v)</td>
<td>$\sin x + b \cos x$</td>
<td>$\theta = r \cos \alpha$, $b = r \sin \alpha$</td>
</tr>
<tr>
<td>(vi)</td>
<td>$\sqrt{x - \alpha}$ and $\sqrt{\beta - x}$</td>
<td>$x = \alpha \sin^2 \theta + \beta \cos^2 \theta$</td>
</tr>
<tr>
<td>(vii)</td>
<td>$\sqrt{2ax - x^2}$</td>
<td>$x = a (1 - \cos \theta)$</td>
</tr>
</tbody>
</table>

3. **Differentiation of Implicit Functions**

If $f(x, y) = 0$, differentiate with respect to $x$ and collect the terms containing $dy / dx$ at one side and find $dy / dx$.

Shortcut for Implicit Functions For Implicit function, put $d / dx \{f(x, y)\} = - \frac{\partial f / \partial x}{\partial f / \partial y}$, where $\partial f / \partial x$ is a partial differential of given function with respect to $x$ and $\partial f / \partial y$ means Partial differential of given function with respect to $y$.

4. **Differentiation of Parametric Functions**

If $x = f(t)$, $y = g(t)$, where $t$ is parameter, then

$$\frac{dy}{dx} = \frac{(dy / dt)}{(dx / dt)} = \frac{d}{dt} \frac{g(t)}{d}{dt} f(t) = g'(t) / f'(t)$$

5. **Differential Coefficient Using Inverse Trigonometrical Substitutions**

Sometimes the given function can be deducted with the help of inverse Trigonometrical substitution and then to find the differential coefficient is very easy.
Logarithmic Differentiation Function

(i) If a function is the product and quotient of functions such as \( y = \frac{f_1(x)}{f_2(x)} \), we first take algorithm and then differentiate.

(ii) If a function is in the form of exponent of a function over another function such as \( \frac{f(x)}{g(x)} \), we first take logarithm and then differentiate.

Differentiation of a Function with Respect to Another Function

Let \( y = f(x) \) and \( z = g(x) \), then the differentiation of \( y \) with respect to \( z \) is

\[
\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dz}{dx}
\]

Successive Differentiations

If the function \( y = f(x) \) be differentiated with respect to \( x \), then the result \( \frac{dy}{dx} \) or \( f'(x) \), so obtained is a function of \( x \) (may be a constant).
Hence, dy / dx can again be differentiated with respect of x.

The differential coefficient of dy / dx with respect to x is written as d /dx (dy / dx) = d²y / dx² or f’ (x). Again, the differential coefficient of d²y / dx² with respect to x is written as

d / dx (d²y / dx²) = d³y / dx³ or f”(x)……

Here, dy / dx, d²y / dx², d³y / dx³,… are respectively known as first, second, third, … order differential coefficients of y with respect to x. These alternatively denoted by f’ (x), f” (x), f”’ (x), … or y₁, y₂, y₃….., respectively.

Note dy / dx = (dy / dθ) / (dx / dθ) but d²y / dx² ≠ (d²y / dθ²) / (d²x / dθ²)

**Leibnitz Theorem**

If u and v are functions of x such that their nth derivative exist, then

\[ D^n(u \cdot v) = nC_0(D^n u)v + nC_1(D^{n-1}u)(Dv) + nC_2(D^{n-2}u)(D^2v) + \ldots + nC_rD^{n-r}u \cdot D^r v + \ldots + nC_n(D^n v) \]

**nth Derivative of Some Functions**

(i) \[ \frac{d^n}{dx^n} [\sin (ax + b)] = a^n \sin \left( \frac{n\pi}{2} + ax + b \right) \]

(ii) \[ \frac{d^n}{dx^n} [\cos(ax + b)] = a^n \cos \left( \frac{n\pi}{2} + ax + b \right) \]

(iii) \[ \frac{d^n}{dx^n} (ax + b)^m = \frac{m!}{(m-n)!} a^n (ax + b)^{m-n} \]

(iv) \[ \frac{d^n}{dx^n} [\log(ax + b)] = \frac{(-1)^{n-1}(n-1)!a^n}{(ax + b)^n} \]

(v) \[ \frac{d^n}{dx^n} (e^{ax}) = a^n e^{ax} \]

(vi) \[ \frac{d^n}{dx^n} (ax^n) = a^n (\log a)^n \]
(vii) (a) \( \frac{d^n}{dx^n}[e^{ax}\sin (bx+c)] = r^n e^{ax} \sin (bx+c+n\phi) \)

\( \phi = \tan^{-1}\left(\frac{b}{a}\right) \)

(b) \( \frac{d^n}{dx^n}[e^{ax}\cos (bx+c)] = r^n e^{ax} \cos (bx+c+n\phi) \)

where, \( r = \sqrt{a^2 + b^2} \)

Derivatives of Special Types of Functions

(i) If \( y = f(x)^{g(x)} \), then \( \frac{dy}{dx} = \frac{y^2 f''(x)}{f(x)[1 - y \log f(x)]} \)

(ii) If \( e^{g(x)} - e^{-g(x)} = 2f(x) \), then \( \frac{dy}{dx} = \frac{f'(x)}{g'(y)} \frac{1}{\sqrt{1 + [f(x)]^2}} \)

(iii) If \( y = \frac{1 + g(x)}{1 - g(x)} \), then \( \frac{dy}{dx} = \frac{g'(x)}{1 - g(x)} \frac{1 - g(x)}{1 + g(x)} \)

(iv) If \( y = \sqrt{f(x) + f(x) + \ldots} \), then \( \frac{dy}{dx} = \frac{f'(x)}{2y - 1} \)

(v) If \( \{f(x)\}^{g(y)} = a^{f(x)} - g(y) \), then \( \frac{dy}{dx} = \frac{f'(x) \log f(x)}{g'(y)[1 + \log f(x)]^2} \)

(vi) If \( \{f(x)\}^{g(y)} = \{g(y)\}^{f(x)} \), then

\[ \frac{dy}{dx} = \frac{g'(y)}{f(x)} \left( \frac{f'(x) \log g(y) - g'(y)}{g'(y) \log f(x)} \right) \]

(vii) **Differentiation of a Determinant**

If \( y = \begin{vmatrix} p & q & r \\ u & v & w \\ l & m & n \end{vmatrix} \), then

\[ \frac{dy}{dx} = \begin{vmatrix} \frac{dp}{dx} & \frac{dq}{dx} & \frac{dr}{dx} \\ \frac{du}{dx} & \frac{dv}{dx} & \frac{dw}{dx} \\ \frac{dl}{dx} & \frac{dm}{dx} & \frac{dn}{dx} \end{vmatrix} + \begin{vmatrix} p & q & r \\ u & v & w \\ l & m & n \end{vmatrix} + \frac{dp}{dx} + \frac{dq}{dx} + \frac{dr}{dx} \]

(viii) **Differentiation of Integrable Functions** If \( g_1(x) \) and \( g_2(x) \) are defined in \([a, b]\), Differentiable at \( x \in [a, b] \) and \( f(t) \) is continuous for \( g_1(a) \leq f(t) \leq g_2(b) \), then

\[ \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \]

and \( \frac{df}{dy} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k} \)

Partial Differentiation
The partial differential coefficient of $f(x, y)$ with respect to $x$ is the ordinary differential coefficient of $f(x, y)$ when $y$ is regarded as a constant. It is written as $\frac{\partial f}{\partial x}$ or $D_x f$ or $f_x$.

$$\frac{d}{dx} \left[ f(G_2(x)) \right] = f'(G_2(x)) \frac{d}{dx} [G_2(x)] - f[G_2(x)] \frac{d}{dx} [G_2(x)]$$

E.g., If $z = f(x, y) = x^4 + y^4 + 3xy^2 + x^4y + x + 2y$

Then, $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or $f_x = 4x^3 + 3y^2 + 2xy + 1$ (here, $y$ is consider as constant)

$\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or $f_y = 4y^3 + 6xy + x^2 + 2$ (here, $x$ is consider as constant)

**Higher Partial Derivatives**

Let $f(x, y)$ be a function of two variables such that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ both exist.

(i) The partial derivative of $\frac{\partial f}{\partial y}$ w.r.t. ‘$x$’ is denoted by $\frac{\partial^2 f}{\partial x^2}$ or $f_{xx}$.

(ii) The partial derivative of $\frac{\partial f}{\partial y}$ w.r.t. ‘$y$’ is denoted by $\frac{\partial^2 f}{\partial y^2}$ or $f_{yy}$.

(iii) The partial derivative of $\frac{\partial f}{\partial x}$ w.r.t. ‘$y$’ is denoted by $\frac{\partial^2 f}{\partial y \partial x}$ or $f_{yx}$.

(iv) The partial derivative of $\frac{\partial f}{\partial x}$ w.r.t. ‘$x$’ is denoted by $\frac{\partial^2 f}{\partial y \partial x}$ or $f_{xy}$.

Note $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

These four are second order partial derivatives.

**Euler’s Theorem on Homogeneous Function**

If $f(x, y)$ be a homogeneous function in $x, y$ of degree $n$, then

$x \left( \frac{\partial f}{\partial x} \right) + y \left( \frac{\partial f}{\partial y} \right) = nf$

**Deduction Form of Euler’s Theorem**

If $f(x, y)$ is a homogeneous function in $x, y$ of degree $n$, then

(i) $x \left( \frac{\partial^2 f}{\partial x^2} \right) + y \left( \frac{\partial^2 f}{\partial x \partial y} \right) = (n - 1) \frac{\partial f}{\partial x}$

(ii) $x \left( \frac{\partial^2 f}{\partial y \partial x} \right) + y \left( \frac{\partial^2 f}{\partial y^2} \right) = (n - 1) \frac{\partial f}{\partial y}$

(iii) $x^2 \left( \frac{\partial^2 f}{\partial x^2} \right) + 2xy \left( \frac{\partial^2 f}{\partial x \partial y} \right) + y^2 \left( \frac{\partial^2 f}{\partial y^2} \right) = n(n - 1) f(x, y)$

**Important Points to be Remembered**

[www.ncerthelp.com](http://www.ncerthelp.com) (Visit for all ncert solutions in text and videos, CBSE syllabus, note and many more)
If \( \alpha \) is \( m \) times repeated root of the equation \( f(x) = 0 \), then \( f(x) \) can be written as

\[
f(x) = (x - \alpha)^m g(x), \text{ where } g(\alpha) \neq 0.
\]

From the above equation, we can see that

\[
f(\alpha) = 0, f'(\alpha) = 0, f''(\alpha) = 0, \ldots , f^{(m-1)}(\alpha) = 0.
\]

Hence, we have the following proposition

\[
f(\alpha) = 0, f'(\alpha) = 0, f''(\alpha) = 0, \ldots , f^{(m-1)}(\alpha) = 0.
\]

Therefore, \( \alpha \) is \( m \) times repeated root of the equation \( f(x) = 0 \).