# Maths Class 11 Chapter 5 Part -1 Quadratic equations

1. **Real Polynomial:** Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and x is a real variable. Then, $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is called a real polynomial of real variable x with real coefficients.

2. **Complex Polynomial:** If $a_0, a_1, a_2, \ldots, a_n$ be complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n$ is called a complex polynomial or a polynomial of complex variable with complex coefficients.

3. **Degree of a Polynomial:** A polynomial $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$ , real or complex is a polynomial of degree n , if $a_n \neq 0$.

4. **Polynomial Equation:** If f(x) is a polynomial, real or complex, then f(x) = 0 is called a polynomial equation. If f(x) is a polynomial of second degree, then f(x) = 0 is called a quadratic equation .

**Quadratic Equation:** A polynomial of second degree is called a quadratic polynomial. Polynomials of degree three and four are known as cubic and biquadratic polynomials respectively. A quadratic polynomial f(x) when equated to zero is called quadratic equation. i.e., $ax^2 + bx + c = 0$ where $a \neq 0$.

**Roots of a Quadratic Equation:** The values of variable x .which satisfy the quadratic equation is called roots of quadratic equation.

**Important Points to be Remembered**

- An equation of degree n has n roots, real or imaginary .
- Surd and imaginary roots always occur in pairs of a polynomial equation with real coefficients i.e., if $(\sqrt{2} + \sqrt{3}i)$ is a root of an equation, then' $(\sqrt{2} - \sqrt{3}i)$ is also its root. .
- An odd degree equation has at least one real root whose sign is opposite to that of its last' term (constant term), provided that the coefficient of highest degree term is positive.
- Every equation of an even degree whose constant term is negative and the coefficient of highest degree term is positive has at least two real roots, one positive and one negative.
- If an equation has only one change of sign it has one positive root.
- If all the terms of an equation are positive and the equation involves odd powers of x, then all its roots are complex.

**Solution of Quadratic Equation**

1.**Factorization Method:** Let $ax^2 + bx + c = \alpha(x - \alpha)(x - \beta) = O$. Then, $x = \alpha$ and $x = \beta$ will satisfy the given equation.

2. **Direct Formula:** Quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ has two roots, given by

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}, \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\alpha = \frac{-b+\sqrt{D}}{2a}, \beta = \frac{-b-\sqrt{D}}{2a}$$

where $D = \Delta = b^2 - 4ac$ is called discriminant of the equation .

Above formulas also known as Sridharacharya formula.

**Nature of Roots**

Let quadratic equation be $ax^2 + bx + c = 0$, whose discriminant is D.

(i) For $ax^2 + bx + c = 0$; a, b , C ∈ R and a ≠ 0, if

(a) D < => Complex roots

(b) D > 0 => Real and distinct roots

(c) D = 0 => Real and equal roots as $\alpha = \beta = -b/2a$

(ii) If a, b, C ∈ Q, a ≠ 0, then

(a) If D > 0 and D is a perfect square => Roots are unequal and rational.

(b) If D > 0, a = 1; b, c ∈ I and D is a perfect square. => Roots are integral. .

(c) If D > and D is not a perfect square. => Roots are irrational and unequal.

(iii) **Conjugate Roots** The irrational and complex roots of a quadratic equation always occur in pairs. Therefore,

(a) If one root be $\alpha + i\beta$, then other root will be $\alpha - i\beta$.

(b) If one root be $\alpha + \sqrt{\beta}$, then other root will be $\alpha - \sqrt{\beta}$.

(iv) If D, and D2 be the discriminants of two quadratic equations, then

(a) If $D_1 + D_2 \geq 0$, then At least one of $D_1$ and $D_2 \geq 0$ If $D_1 < 0$, then $D_2 > 0$ ,

(b) If $D_1 + D_2 < 0$, then At least one of $D_1$ and $D_2 < 0$ If $D_1 > 0$, then $D_2 < 0$

**Roots Under Particular Conditions**

For the quadratic equation $ax^2 + bx + e = 0$.

(i) If b = 0 => Roots are real/complex as (c < 0/c > 0) and equal in magnitude but of opposite sign.

(ii) If c = 0 => One roots is zero, other is – b / a.

(iii) If b = C = 0 => Both roots are zero.

(iv) If a = c => Roots are reciprocal to each other.

(v) If a > 0, c < 0, a < 0, c > 0} => Roots are of opposite sign.

(vi) If a > 0, b > 0, c > 0, a < 0, b < 0, c < 0} => Both roots are negative, provided D ≥ 0

(vii) If a > 0, b < 0, c > 0, a < 0, b > 0, c < 0} => Both roots are positive, provided D ≥ 0

(viii) If sign of a = sign of b ≠ sign of c => Greater root in magnitude is negative.

(ix) If sign of b = sign of c ≠. sign of a => Greater root in magnitude is positive.

(x) If a + b + c = 0 => One root is 1 and second root is c/a.

**Relation between Roots and Coefficients**

1. **Quadratic Equation:** If roots of quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) are α and β, then Sum of roost = S = α + β = -b/a = – coefficient of x / coefficient of $x^2$ Product of roots = P = α * β = c/a = constant term / coefficient of $x^2$

2. **Cubic Equation:** If α, β and γ are the roots of cubic equation $ax^3 + bx^2 + cx + d = 0$.

Then,

$$\sum \alpha = \alpha + \beta + \gamma = -\frac{b}{a}$$

$$\sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

3. **Biquadratic Equation:** If α, β, γ and δ are the roots of the biquadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$S_1 = \alpha + \beta + \gamma = -\frac{b}{a},$$

$$S_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = (-1)^2\frac{c}{a} = \frac{c}{a}$$

$$S_2 = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{c}{a}$$

$$S_3 = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \alpha\beta\delta = (-1)^3 \frac{d}{a} = -\frac{d}{a}$$

$$S_3 = \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -\frac{d}{a}$$

$$S_4 = \alpha \cdot \beta \cdot \gamma \cdot \delta = (-1)^4 \frac{e}{a} = \frac{e}{a}$$

**Symmetric Roots:** If roots of quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are α and β, then

(i) $(\alpha - \beta) = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \pm \frac{\sqrt{b^2 - 4ac}}{a} = \frac{\pm\sqrt{D}}{a}$

(ii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{b^2 - 2ac}{a^2}$

(iii) $\alpha^2 - \beta^2 = (\alpha + \beta)\sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \pm \frac{b\sqrt{b^2 - 4ac}}{a^2} = \frac{\pm b\sqrt{D}}{a^2}$

(iv) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = -\frac{b(b^2 - 3ac)}{a^3}$

(v) $\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta) = \frac{\pm(b^2 - ac)\sqrt{b^2 - 4ac}}{a^3}$

(vi) $\alpha^4 + \beta^4 = \{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 2\alpha^2\beta^2 = \left(\frac{b^2 - 2ac}{a^2}\right)^2 - \frac{2c^2}{a^2}$

(vii) $\alpha^4 - \beta^4 = (\alpha^2 - \beta^2)(\alpha^2 + \beta^2) = \frac{\pm b(b^2 - 2ac)\sqrt{b^2 - 4ac}}{a^4}$

(viii) $\alpha^2 + \alpha\beta + \beta^2 = (\alpha + \beta)^2 - \alpha\beta = \frac{b^2 - ac}{a^2}$

(ix) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{b^2 - 2ac}{ac}$

(x) $\alpha^2\beta + \beta^2\alpha = \alpha\beta(\alpha + \beta) = -\frac{bc}{a^2}$

(xi) $\left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\beta}{\alpha}\right)^2 = \frac{\alpha^4 + \beta^4}{\alpha^2\beta^2} = \frac{(\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2}{\alpha^2\beta^2} = \frac{b^2D + 2a^2c^2}{a^2c^2}$

**Formation of Polynomial Equation from Given Roots**

If $a_1$, $a_2$ $a_3$,…, an are the roots of an nth degree equation, then the equation is $x^n - S_1X^{n-1} + S_2X^{n-2} - S_3X^{n-3} + \ldots + (\_1)^n S_n = 0$ where $S_n$ denotes the sum of the products of roots taken n at a time.

1. **Quadratic Equation**

If α and β are the roots of 'a quadratic equation, then the equation is $x^2 - S_1X + S_2 = 0$

i.e., $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

2. **Cubic Equation**

If α, β and γ are the roots of cubic equation, then the equation is

$$x^3 - S_1x^2 + S_2x - S_3 = 0$$
$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma = 0$$

3. **Biquadratic Equation**

If α, β, γ and δ are the roots of a biquadratic equation, then the equation is

$$x^4 - S_1x^3 + S_2x^2 - S_3x + S_4 = 0$$
$$x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \beta\gamma + \gamma\delta + \alpha\delta + \beta\delta + \alpha\gamma)x^2 - (\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \gamma\delta\alpha)x + \alpha\beta\gamma\delta = 0$$

**Equation In Terms of the Roots of another Equation**

If α, β are roots of the equation $ax^2 + bx + c = 0$, then the equation whose roots are.

(i) $-\alpha, -\beta \Rightarrow ax^2 - bx + c = 0$ (replace $x$ by $-x$)

(ii) $\alpha^n, \beta^n; n \in N \Rightarrow a(x^{1/n})^2 + b(x^{1/n}) + c = 0$ (replace $x$ by $x^{1/n}$)

(iii) $k\alpha, k\beta \Rightarrow ax^2 + kbx + k^2c = 0$ (replace $x$ by $x/k$)

(iv) $k + \alpha, k + \beta \Rightarrow a(x - k)^2 + b(x - k) + c = 0$ (replace $x$ by $(x - k)$)

(v) $\frac{\alpha}{k}, \frac{\beta}{k} \Rightarrow k^2ax^2 + kbx + c = 0$ (replace $x$ by $kx$)

(vi) $\alpha^{1/n}, \beta^{1/n}; n \in N \Rightarrow a(x^n)^2 + b(x^n) + c = 0$ (replace $x$ by $x^n$)

The quadratic function $f(x) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ is always resolvable into linear factor, iff

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

**Condition for Common Roots in a Quadratic Equation**

1. **Only One Root is Common**

If α be the common root of quadratic equations

$a_1x^2 + b_1x + C_1 = 0$,

and $a_2x^2 + b_2x + C_2 = 0$,

then $a_1a^2 + b_1\alpha + C_1 = 0$,

and $a_2a^2 + b_2\alpha + C_2 = 0$,

**By Cramer's Rule**

$$\frac{\alpha^2}{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}} = \frac{\alpha}{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\alpha = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} = \frac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2}, \alpha \neq 0$$

Hence, the condition for only one root common is

$(c_1a_2 - c_2a_1)_2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$

2. **Both Roots are Common**

The required condition is

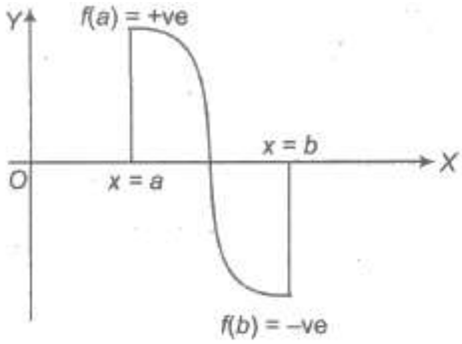
$a_1 / a_2 = b_1 / b_2 = c_1 / c_2$

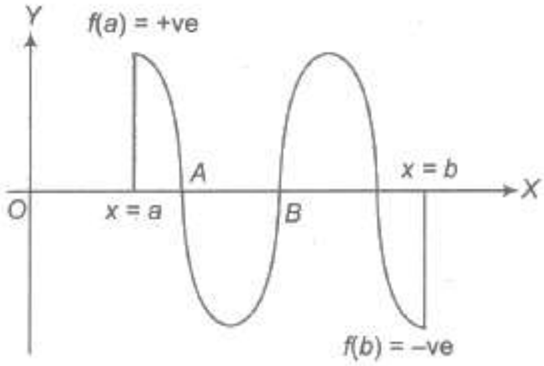
(i) To find the common root of two equations, make the coefficient of second degree term in the two equations equal and subtract. The value of x obtained is the required common root.

(ii) Two different quadratic equations with rational coefficient can not have single common root which is complex or irrational as imaginary and surd roots always occur in pair.

**Properties of Quadratic Equation**

(i) f(a) . f(b) < 0, then at least one or in general odd number of roots of the equation f(x) = 0 lies between a and b.




(ii) f( a) . f( b) > 0, then in general even number of roots of the equation f(x) = 0 lies between a and b or no root exist f(a) = f(b), then there exists a point c between a and b such that f'(c) = 0, a < c < b.

(iii) If the roots of the quadratic equation $a_1x^2 + b_1x + c_1 = 0$, $a_2x^2 + b_2x + c_2 = 0$ are in the ratio (i.e., &alpha$_1$;/$\beta_1$ = &alpha$_2$;/$\beta_2$), then

$b_1^2 / b_2^2 = a_1c_1 / a_2c_2$.

(iv) If one root is k times the other root of the quadratic equation $ax^2 + bx + c = 0$ ,then

$(k + 1)^2 / k = b^2 / ac$

**Quadratic Expression**

An expression of the form $ax^2 + bx + c$, where a, b, c ∈ R and a ≠ 0 is called a quadratic expression in x .

1. **Graph of a Quadratic Expression**

We have

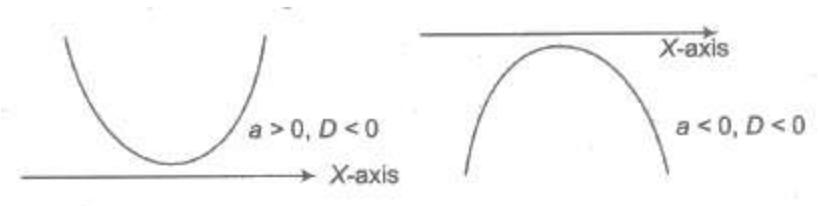
$y = ax^2 + bx + c = f(x)$

$$y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right] \Rightarrow y + \frac{D}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

Let y + D/4a = Y and x + D / 2a = X

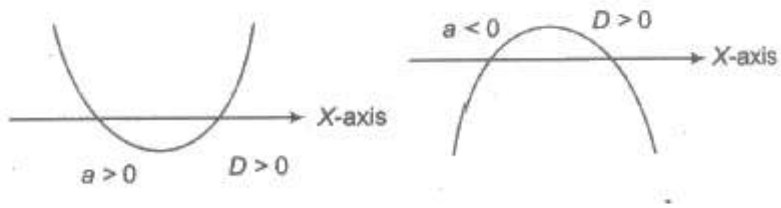
$Y = a * X^2 \Rightarrow X^2 = Y / a$

(i) The graph of the curve y = f(x) is parabolic.
(ii) The axis of parabola is X = 0 or x + b / 2a = 0 i.e., (parallel to Y-axis). •
(iii) If a > 0, then the parabola opens upward.
If a < 0, then the parabola opens downward.

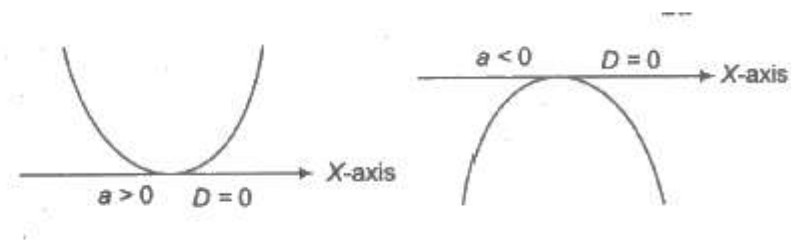



2. **Position of** $y = ax^2 + bx + c$ **with Respect to Axes.**

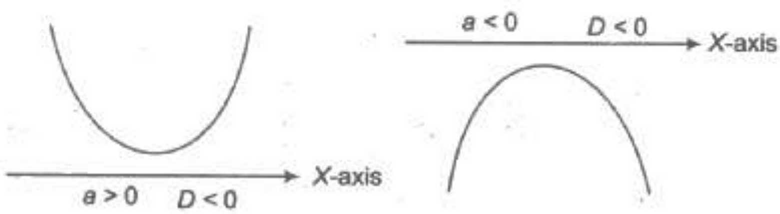
(i) For D > 0, parabola cuts X-axis in two real and distinct points
i.e, x = -b ± √D / 2a




(ii) For D = 0, parabola touch X-axis in one point, x = – b/2a.




(iii) For D < O,parabola does not cut X-axis (i.e., imaginary value of x).



3. **Maximum and Minimum Values of Quadratic Expression**

(i) If $a > 0$, quadratic expression has least value at $x = b / 2a$. This least value is given by $4ac – b^2 / 4a = – D/4a$. But their is no greatest
value.

(ii) If $a < 0$, quadratic expression has greatest value at $x = – b/2a$ . This greatest value is given by $4ac – b^2 / 4a = – D/4a$. But their is no least value.

4. **Sign of Quadratic Expression**

(i) $a > 0$ and $D < 0$, so $f(x) > 0$ for all $x \in R$ i.e., f(x) is positive for all real values of x.

(ii) $a < 0$ and $D < 0$, so $f(x) < 0$ for all $x \in R$ i.e., f(x) is negative for all real values of x.

(iii) $a > 0$ and $D = 0$, so $f(x) \geq 0$ for all $x \in R$ i.e., f(x) is positive for all real values of x except at vertex, where $f(x) = 0$.

(iv) $a < 0$ and $D = 0$, so $f(x) \leq 0$ for all $x \in R$ i.e., f(x) is negative for all real values of x except at vertex, where $f(x) = 0$.

(v) $a > 0$ and $D > 0$
Let f(x) = o have two real roots $\alpha$ and $\beta$ $(\alpha < \beta)$, then $f(x) > 0$ for $x \in (- \infty, \alpha) \cup (\beta,\infty)$ and $f (x) < 0$ for all $x \in (\alpha, \beta)$.

(vi) $a < 0$ and $D > 0$
Let $f(x) = 0$ have two real roots $\alpha$ and $\beta$ $(\alpha < \beta)$. Then, $f(x) < 0$ for all $x \in (- \infty, \alpha) \cup (\beta,\infty)$ and $f(x) > 0$ for all
$x \in (\alpha, \beta)$. ,

5. **Intervals of Roots**

In some problems, we want the roots of the equation $ax^2 + bx + c = 0$ to lie in a given interval. For this we impose conditions on a, b and c.
Since, $a \neq 0$, we can take $f(x) = x^2 + b/a\ x + c/a$.

(i) Both the roots are positive i.e., they lie in $(0,\infty)$, if and only if roots are real, the sum of the roots as well as the product of the roots is positive.

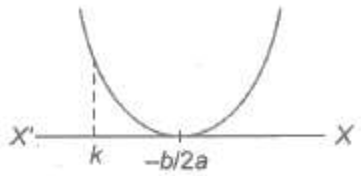
$\alpha + \beta = -b/a > 0$ and $\alpha\beta = c/a > 0$ with $b^2 – 4ac \geq 0$

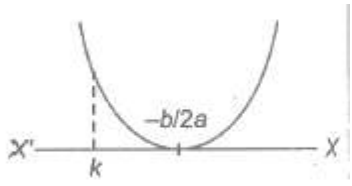
Similarly, both the roots are negative i.e., they lie in $(- \infty,0)$ ifF roots are real, the sum of the roots is negative and the product of the roots is positive.

i.e., $\alpha + \beta = -b/a < 0$ and $\alpha\beta = c/a > 0$ with $b^2 – 4ac \geq 0$

(ii) Both the roots are greater than a given number k, iFf the following conditions are satisfied

$D \geq 0$, $-b/2a > k$ and $f(k) > 0$



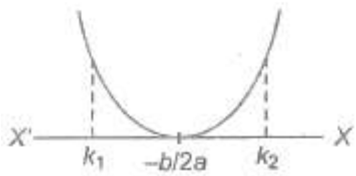



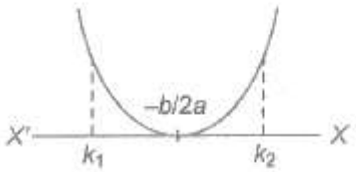

(iii) Both .the roots are less than a given number k, iff the following conditions are satisfied

$D \geq 0$, $-b/2a > k$ and $f(k) > 0$

(iv) Both the roots lie in a' given interval $(k_1, k_2)$, iff the following conditions are satisfied

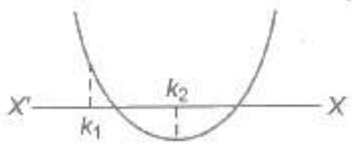
$D \geq 0, k_1 < -b/2a < k_2$ and $f(k_1) > 0$, $f(k_2) > 0$

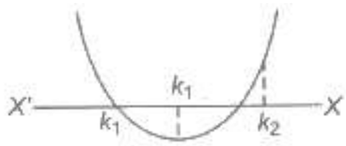



(v) Exactly one of the roots lie in a given interval $(k_1, k_2)$, iff

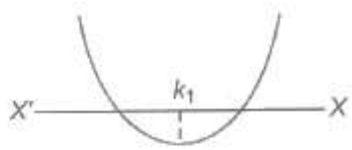
$f(k_1)\, f(k_2) < 0$




(vi) A given number k lies between the roots iff f(k) < O. In particular, the roots of the equation will be of opposite sign, iff 0 lies between the roots.

$\Rightarrow f(0) < 0$




**Wavy Curve Method**

Let $f(x) = (x - a_1)^{k_1} (x - a_2)^{k_2} (x - a_3)^{k_3} \ldots (x - a_{n-1})^{k_{n-1}} (x - a_n)^{k_n}$

where $k_1, k_2, k_3, \ldots, k_n \in N$ and $a_1, a_2, a_3, \ldots, a_n$ are fixed natural numbers satisfying the condition.

$a_1 < a_2 < a_3 < \ldots < a_{n-1} < a_n$..

First we mark the numbers $a_1, a_2, a_3, \ldots, a_n$ on the real axis and the plus sign in the interval of the right of the largest of these numbers, i.e., on the right of $a_n$. If $k_n$ is even, we put plus sign on the left of $a_n$and if $k_n$ is odd, then we put minus sign on the left of $a_n$ In the next interval we put a sign according to the following rule.

When passing through the point $a_{n-1}$ the polynomial f(x) changes sign . if $k_{n-1}$ is an odd number and the polynomial f(x) has same sign if $k_{n-1}$ is an even number. Then, we consider the next interval and put a sign in it using the same rule.

Thus, we consider all the intervals. The solution of $f(x) > 0$ is the union of all interval in which we have put the plus sign and the solution of $f(x) < 0$ is the union of all intervals in which we have put the minus Sign.

**Descarte's Rule of Signs**

The maximum number of positive real roots of a polynomial equation f(x) = 0 is the number of changes of sign from positive to negative and negative to positive in f(x) .

The maximum number of negative real roots of a polynomial equation f(x) = 0 is the number of changes of sign from positive to negative and negative to positive in f(x).

**Rational Algebraic In equations**

(i) **Values of Rational Expression P(x)/Q(x)** for Real Values of x, where **P(x) and Q(x) are Quadratic Expressions** To find the values attained by rational expression of the form $a_1x^2 + b_1x + c_1 / a_2x^2 + b_2x + c_2$

for real values of x.

(a) Equate the given rational expression to y.
(b) Obtain a quadratic equation in x by simplifying the expression,
(c) Obtain the discriminant of the quadratic equation.
(d) Put discriminant $\geq 0$ and solve the in equation for y. The values of y so obtained determines the set of values attained by the given rational expression.

(ii) **Solution of Rational Algebraic In equation** If P(x) and Q(x) are polynomial in x, then the in equation $P(x) / Q(x) > 0$,
$P(x) / Q(x) < 0$, $P(x) / Q(x) \geq 0$ and $P(x) / Q(x) \leq 0$ are known as rational algebraic in equations.

To solve these in equations we use the sign method as

(a) Obtain P(x) and Q(x).
(b) Factorize P(x) and Q(x) into linear factors.
(c) Make the coefficient of x positive in all factors.
(d) Obtain critical points by equating all factors to zero.
(e) Plot the critical points on the number line. If these are n critical points, they divide the number line into (n + 1) regions.
(f) In the right most region the expression P(x) / Q(x) bears positive sign and in other region the expression bears positive and negative signs depending on the exponents of the factors .

**Lagrange's identity**

If $a_1, a_2, a_3, b_1, b_2, b_3 \neq R$, then

$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$

$= (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2$

**Algebraic Interpretation of Rolle's Theorem**

Let f (x) be a polynomial having α and β as its roots such that $\alpha < \beta$, $f(\alpha) = f(\beta) = 0$.Also, a polynomial function is everywhere continuous and differentiable, then there exist $\theta \in (\alpha, \beta)$ such that $f'(\theta) = 0$. Algebraically, we can say between any two zeros of a polynomial f(x) there is always a derivative f' (x) = 0.

**Equation and In equation Containing Absolute Value**

1. **Equation Containing Absolute Value**

By definition, $|x| = x$, if $x \geq 0$ OR $-x$, if $x < 0$

If $|f(x) + g(x)| = |f(x)| + g(x)|$, then it is equivalent to the system $f(x) . g(x) \geq 0$.

If $|f(x) - g(x)| = |f(x)| - g(x)|$, then it is equivalent to the system $f(x) . g(x) \leq 0$.

2.**In equation Containing Absolute Value**

(i) $|x| < a \Rightarrow - a < x < a$ $(a > 0)$
(ii) $|x| \leq a \Rightarrow - a \leq x \leq a$
(iii) $|x| > a \Rightarrow x < - a$ or $x > a$
(iv) $|x| \geq a \Rightarrow$ x le; – a or $x \geq a$

3. **Absolute Value of Real Number**

$|x| = -x, x < 0$ OR $+x, x \geq 0$

(i) $|xy| = |x||y|$
(ii) $|x / y| = |x| / |y|$
(iii) $|x|^2 = x^2$
(iv) $|x| \geq x$
(v) $|x + y| \leq |x| + |y|$
Equality hold when x and y same sign.
(vi) $|x - y| \geq ||x| - |y||$

## Inequalities

Let a and b be real numbers. If a – b is negative, we say that a is less than b (a < b) and if a – b is positive, then a is greater than b (a > b).

## Important Points to be Remembered

(i) If a > b and b > c, then a > c. Generally, if $a_1 > a_2$, $a_2 > a_3, \ldots, a_{n-1} > a_n$, then $a_1 > a_n$.

**(ii)** If $a > b$, then $a \pm c > b \pm c, \forall c \in R$

**(iii)** (a) If $a > b$ and $m > 0$, $am > bm, \frac{a}{m} > \frac{b}{m}$

(b) If $a > b$ and $m < 0$, $bm < am, \frac{b}{m} < \frac{a}{m}$

**(iv)** If $a > b > 0$, then

(a) $a^2 > b^2$ (b) $|a| > |b|$ (c) $\frac{1}{a} < \frac{1}{b}$

**(v)** If $a < b < 0$, then

(a) $a^2 > b^2$ (b) $|a| > |b|$ (c) $\frac{1}{a} > \frac{1}{b}$

**(vi)** If $a < 0 < b$, then

(a) $a^2 > b^2$, if $|a| > |b|$
(b) $a^2 < b^2$, if $|a| < |b|$

(vii) If a < x < b and a, b are positive real numbers then $a^2 < x^2 < b^2$

**(viii)** If $a < x < b$ and $a$ is negative number and $b$ is positive number, then

(a) $0 \le x^2 < b^2$, if $|b| > |a|$

(b) $0 \le x^2 \le b^2$, if $|a| > |b|$

**(ix)** If $\frac{a}{b} > 0$, then

(a) $a > 0$, if $b > 0$

(b) $a < 0$, if $b < 0$

**(x)** If $a_i > b_i > 0$, where $i = 1, 2, 3, \dots, n$, then

$$a_1 a_2 a_3 \dots a_n > b_1 b_2 b_3 \dots b_n$$

**(xi)** If $|x| < a$ and

(a) if $a$ is positive, then $-a < x < a$.

(b) if $a$ is negative, then $x \in \phi$

**(xii)** If $a_i > b_i$, where $i = 1, 2, 3, \dots, n$, then

$a_1 + a_2 + a_3 + \dots + a_n > b_1 + b_2 + \dots + b_n$

**(xiii)** If $0 < a < 1$ and $n$ is a positive rational number, then

(a) $0 < a^n < 1$ (b) $a^{-n} > 1$

**Important Inequality**

1. **Arithmetico-Geometric and Harmonic Mean Inequality**

(i) If a, b > 0 and a ≠ b, then

$$\frac{a+b}{2} > \sqrt{ab} > \frac{2}{(1/a)+(1/b)}$$

(ii) if $a_i > 0$, where i = 1,2,3,…,n, then

$$\frac{a_1 + a_2 + \dots + a_n}{n} \ge (a_1 \cdot a_2 \cdot \dots \cdot a_n)^{1/n}$$

$$\ge \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}}$$

(iii) If $a_1, a_2, \dots, a_n$ are n positive real numbers and $m_1, m_2, \dots, m_n$ are n positive rational numbers, then

$$\frac{m_1 a_1 + m_2 a_2 + \dots + m_n a_n}{m_1 + m_2 + \dots + m_n} > \left(a_1^{m_1} \cdot a_2^{m_2} \cdot \dots \cdot a_n^{m_n}\right)^{\frac{1}{m_1 + m_2 + \dots + m_n}}$$

i.e., Weighted AM > Weighted GM

(iv) If $a_1, a_2, \ldots, a_n$ are n positive distinct real numbers, then

(a)
$$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} > \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m \text{ if } m < 0 \text{ or } m > 1$$

(b)
$$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} < \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m, \text{ if } 0 < m < 1$$

(c) If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are rational numbers and M is a rational number, then

$$\frac{b_1 a_1^m + b_2 a_2^m + \ldots + b_n a_n^m}{b_1 + b_2 + \ldots + b_n} > \left(\frac{b_1 a_1 + b_2 a_2 + \ldots + b_n a_n}{b_1 + b_2 + \ldots + b_n}\right)^m, \text{if } 0 < m < 1$$

$$\frac{b_1 a_1^m + b_2 a_2^m + \ldots + b_n a_n^m}{b_1 + b_2 + \ldots + b_n} < \left(\frac{b_1 a_1 + b_2 a_2 + \ldots + b_n a_n}{b_1 + b_2 + \ldots + b_n}\right)^m,$$

(d) $\text{if } 0 < m < 1$

(v) If $a_1, a_2, a_3, \ldots, a_n$ are distinct positive real numbers and p, ,q, r are natural numbers, then

$$\frac{a_1^{p+q+r} + a_2^{p+q+r} + \ldots + a_n^{p+q+r}}{n} > \left(\frac{a_1^p + a_2^p + \ldots + a_n^p}{n}\right) \left(\frac{a_1^q + a_2^q + \ldots + a_n^q}{n}\right) \left(\frac{a_1^r + a_2^r + \ldots + a_n^r}{n}\right)$$

2. **Cauchy – Schwartz's inequality**

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, such that

$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \ldots, a_n^2) * (b_1^2 + b_2^2 + \ldots, b_n^2)$

Equality holds, iff $a_1 / b_1 = a_2 / b_2 = a_n / b_n$

3. **Tchebychef's Inequality**

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, such that

(i) If $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq b_3 \leq \ldots \leq b_n$, then

$n(a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n) \geq (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n)$

(ii) If If $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n$ and $b_1 \geq b_2 \geq b_3 \geq \ldots \geq b_n$, then

$n(a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n) \leq (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n)$

4. **Weierstrass Inequality**

(i) If $a_1, a_2, \ldots, a_n$ are real positive numbers, then for $n \geq 2$

$(1 + a_1)(1 + a_2) \ldots (1 + a_n) > 1 + a_1 + a_2 + \ldots + a_n$

(ii) If $a_1, a_2, \ldots, a_n$ are real positive numbers, then

$(1 - a_1)(1 - a_2) \ldots (1 - a_n) > 1 - a_1 - a_2 - \ldots - a_n$

5. **Logarithm Inequality**

(i) (a) When $y > 1$ and $\log_y x > z \Rightarrow x > y^z$

(b) When $y > 1$ and $\log_y x < z \Rightarrow 0 < x < y^z$

(ii) (a) When $0 < y < 1$ and $\log_y x > z \Rightarrow 0 < x < y^z$

(b) hen $0 < y < 1$ and $\log_y x < z \Rightarrow x > y^z$

**Application of Inequalities to Find the Greatest and Least Values**

(i) If $x_l, x_2, \ldots, x_n$ are n positive variables such that $x_l + x_2 + \ldots + x_n = c$ (constant), then the product $x_l * x_2 * \ldots * x_n$ is greatest when $x_1 = x_2 = \ldots = x_n = c/n$ and the greatest value is $(c/n)^n$.

(ii) If $x_l, x_2, \ldots, x_n$ are positive variables such that $x_l, x_2, \ldots, x_n = c$ (constant), then the sum $x_l + x_2 + \ldots + x_n$ is least when $x_1 = x_2 = \ldots = x_n = c^{1/n}$ and the least value of the sum is $n(c^{1/n})$.

(iii) If $x_l, x_2, \ldots, x_n$ are variables and $m_l, m_2, \ldots, m_n$ are positive real number such that $x_l + x_2 + \ldots + x_n = c$ (constant), then $x_l^{m_l} * x_2^{m_2} * \ldots * x_n^{m_n}$ is greatest, when

$x_l / m_l = x_2 / m_2 = \ldots = x_n / m_n$

$= x_l + x_2 + \ldots + x_n / m_l + m_2 + \ldots + m_n$